# Analytic solution of flat-top Gaussian and Laguerre–Gaussian laser field components

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We generalized the nonparaxial field components of Laguerre–Gaussian and flattened Gaussian beams obtained using the angular spectrum method to include symmetric radial and angular expansions and simplified them using an approximate evaluation of the integral equations for the field components. These field components possess series expressions in orders of a natural expansion parameter, which clarifies the physical interpretation of the series expansion. A connection between Laguerre–Gaussian and flat-top Gaussian profiles is obtained. © 2010 Optical Society of America

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Although the paraxial approximation is of central importance to understand far-field electromagnetic field effects, it is often necessary to employ extensions of this approach when the ratio of the wavelength to the beam diameter is close to unity. Nonnegligible longitudinal contributions must then be evaluated. The realization that longitudinal contributions arise naturally and consistently from an order expansion of the source-free Maxwell equations was emphasized by Lax *et al.* [1]. This early work provided the context for most of the subsequent extensions of the paraxial approximation to include longitudinal field contributions. For example, the radial emission of electrons from an intense laser beam requires such a treatment, as previously investigated by Cicchitelli *et al.* [2] and Quesnel and Mora [3]. Generalization of the simple Gaussian shape of the beam in an aperture to radially symmetric field distributions was accomplished by Sepke and Umstadter in a series of publications that emphasized applications to flattened Gaussian modes, annular Gaussian modes, and tightly focused spot sizes ([4] and references therein). Previous work on Laguerre–Gaussian beams and corrections to the paraxial approximation using the Felsen complex source approach has also been reported by Bandres and Gutiérrez-Vega [5] and Yan and Yao [6]. Zhou [7] has also used the angular spectrum method to examine Laguerre–Gaussian beams. The development presented here attempts to recast these previous derivations in a simplified form.

More specifically, the standard development of a general flat-top Gaussian or Laguerre–Gaussian field in an aperture is generalized to more flexible radial and angular forms, similar to those of Sepke and Umstadter [4]. Simultaneously, simplification is achieved by splitting the domain of the amplitude functions as in Cicchitelli *et al.* [2] and Agrawal and Pattanayak [8]. Deviations from a pure paraxial approximation may be explicitly tracked in this formalism in terms of a natural expansion parameter, $f$, which is the ratio of the wavelength to the beam waist [1]. The resulting expressions for the vector field components are obtained as series in terms of Bessel functions and Laguerre polynomials and are thus numerically more tractable than previously reported series expansions for flattened Gaussian distributions. To our knowledge, no such complete analytic solution currently exists for Laguerre–Gaussian beams.

The present formalism developed here follows directly from expressions appearing in Cicchitelli *et al.* [2], Agrawal and Pattanayak [8], and extended in Sepke and Umstadter [4]. These earlier derivations relied upon the angular spectrum solution to the vector field components in an aperture [9,10] (see also the Bouwkamp review [11]). This technique is also employed in the following development.

Consider the general expression describing a flattened Gaussian or Laguerre–Gaussian distribution for the transverse field component in the aperture ($z = 0$), where without loss of generality $E_y = 0$ [4]:

$$E_x(x,y,0) = \sum_{n=0}^{N}\sum_{l=-n}^{n} a_{n,l}\left(\frac{r}{\sqrt{2}w_0}\right)^n \exp\left[-\frac{r^2}{2w_0^2}\right]\exp[il\phi], \tag{1}$$

and the coefficients $a_{n,l}$ describe the structure of the distribution. There is an additional requirement that $l$ is only even or odd in agreement with $n$ being even or odd, i.e., $a_{\mathrm{even,odd}} = a_{\mathrm{odd,even}} = 0$.

One could obviously rewrite the polynomial portion of this expansion in terms of Laguerre polynomials of the squared radius, $L_n(\frac{r^2}{2w_0^2})$, as is typical in previously published results, but the chosen form simplifies the derivation and result. At this point we observe that flat-top Gaussian profiles are Laguerre–Gaussian profiles that only allow $n$ even and $l = 0$, and, as we will show, a single derivation provides solutions to both forms of profiles in the aperture. This field component, as described by the angular spectrum method, can be characterized over all space by the generalized amplitude function, $A_x(p,q)$:

$$A_x(p,q) = \left(\frac{k}{2\pi}\right)^2 \int \sum_{n,l} a_{n,l}\left(\frac{r}{\sqrt{2}w_0}\right)^n \exp\left[-\frac{r^2}{2w_0^2}\right] \times \exp[il\phi]\exp[-ik\mathbf{b}\cdot\mathbf{r}]r\mathrm{d}r\mathrm{d}\phi, \tag{2}$$

where $\mathbf{b}\cdot\mathbf{r}$ represents the usual angular spectrum variables $px + qy = br\cos\phi$, defining $b^2 = p^2 + q^2$. The angular integral can be evaluated immediately leaving

$$A_x(p,q) = \frac{k^2}{2\pi}\sum_{n,l} a_{n,l}(i)^l \int_0^\infty \left(\frac{r}{\sqrt{2}w_0}\right)^n \exp\left[-\frac{r^2}{2w_0^2}\right] \times J_l(kbr)r\mathrm{d}r. \quad (3)$$

This integral has a well-known solution [12] when $n - l \geq 0$, which can be written in terms of exponentials and generalized Laguerre polynomials:

$$A_x(p,q) = \frac{1}{2\pi f^2}\sum_{n,l} a_{n,l}(i)^l \left(\frac{b}{\sqrt{2}f}\right)^l c_-! \exp\left[-\frac{b^2}{2f^2}\right] \times L_{c_-}^{(l)}\left(\frac{b^2}{2f^2}\right), \quad (4)$$

where $f = \frac{1}{kw_0} = \frac{\lambda}{2\pi w_0}$ is the natural expansion factor appearing in Louisell *et al.* [1] and $c_\pm = \frac{n \pm l}{2}$, which are always nonnegative integers given the restrictions on $n$ and $l$.

Although this expansion holds for all real $p$ and $q$, a useful approximation is to ignore evanescent waves corresponding to the condition $b^2 > 1$, such that $A_x(p,q) = 0$ for these values. Thus, the amplitude function is nonvanishing only for the range $0 \leq b^2 \leq 1$. The complete spatial dependence of the vector component $E_x$ becomes

$$E_x(x,y,z) = \frac{1}{2\pi f^2}\sum_{n,l} a_{n,l}(i)^l c_-! \int_0^1 \int_0^{2\pi} \left(\frac{b}{\sqrt{2}f}\right)^l \times \exp\left[-\frac{b^2}{2f^2}\right] L_{c_-}^{(l)}\left(\frac{b^2}{2f^2}\right) \times \exp[ik\mathbf{b}\cdot\mathbf{r}] \exp\left[ikz\sqrt{1-b^2}\right] b\mathrm{d}b\mathrm{d}\theta. \quad (5)$$

Using the multiplication relations for Bessel and modified Bessel functions following [2,8], the last exponential term can be expressed as an exact summation in the form

$$\exp\left[ikz\left(\sqrt{1-b^2}\right)\right] = \sum_{m=0}^\infty \frac{1}{m!}\left(\frac{b^2}{2}\right)^m (kz)^{m+1} h_{m-1}^{(1)}(kz), \quad (6)$$

for $b^2 < 1$ and where $h_{m-1}^{(1)}(kz)$ is the $(m-1)$st-order spherical Bessel function of the third kind. Evaluating the angular integral, the field component now appears as

$$E_x(x,y,z) = \sum_{m=0}^\infty \frac{1}{m!}(kz)^{m+1} h_{m-1}^{(1)}(kz) \times \sum_{n,l} a_{n,l}(i)^l c_-! I_{m,n,l}(r), \quad (7)$$

where

$$I_{m,n,l}(r) = \frac{1}{f^{l+2}}\int_0^1 \left(\frac{b}{\sqrt{2}}\right)^{2m+l} \exp\left[-\frac{b^2}{2f^2}\right] \times L_{c_-}^{(l)}\left(\frac{b^2}{2f^2}\right) J_0(bkr) b\mathrm{d}b. \quad (8)$$

This last integral may now be expressed as a difference of two infinite-range integrals

$$I_{m,n,l}(r) = \frac{1}{f^{l+2}}\int_0^\infty \left(\frac{b}{\sqrt{2}}\right)^{2m+l} \exp\left[-\frac{b^2}{2f^2}\right] \times L_{c_-}^{(l)}\left(\frac{b^2}{2f^2}\right) J_0(bkr) b\mathrm{d}b - \frac{1}{f^{l+2}}\int_1^\infty \left(\frac{b}{\sqrt{2}}\right)^{2m+l} \exp\left[-\frac{b^2}{2f^2}\right] \times L_{c_-}^{(l)}\left(\frac{b^2}{2f^2}\right) J_0(bkr) b\mathrm{d}b, \quad (9)$$

each of which converges absolutely for fixed $m$. Because the large $b$ dependence is controlled by the Gaussian exponential term, the second integral may be neglected for $f < 0.4$ [2,8] when $N = 0$ (and, hence, $l = 0$). A more general error bound for this integral is presented below.

After discarding the second integral and substituting the explicit expansion of the associated Laguerre polynomial into $I_{m,n,l}(r)$ yields the analytical form [12]

$$I_{m,n,l}(r) = f^{2m}\sum_{j=0}^{c_-}(-1)^j \frac{c_+!\Gamma\left(m+j+\frac{l}{2}+1\right)}{j!(c_- - j)!(l+j)!} \times \exp\left[-\frac{r^2}{2w_0^2}\right] L_{m+j+\frac{l}{2}}\left(\frac{r^2}{2w_0^2}\right). \quad (10)$$

With this result in hand, the $E_x$ field component has the simplified form,

$$E_x(x,y,z) = \sum_{m=0}^\infty f^{2m}(kz)^{m+1} h_{m-1}^{(1)}(kz) \times \sum_{n=0}^N \sum_{l=-n}^n a_{n,l} \times \sum_{j=0}^{c_-} i^l (-1)^j C_{m,n,l,j} \times \exp\left[-\frac{r^2}{2w_0^2}\right] L_{m+j+\frac{l}{2}}\left(\frac{r^2}{2w_0^2}\right), \quad (11)$$

with

$$C_{m,n,l,j} = \frac{c_+! c_-! \Gamma\left(m+j+\frac{l}{2}+1\right)}{m! j! (c_- - j)! (l+j)!}. \quad (12)$$

This last expression represents a new, analytic solution to Laguerre–Gaussian beams beyond previously known results [5–7]. Additionally, this is a significant simplification of earlier expressions of flattened Gaussian beams [4] because the integral may be evaluated in terms of Bessel and Laguerre functions rather than more complicated special functions such as Gegenbauer polynomials. Finally, the dependence of the electric field component on the expansion parameter, $f$, is explicit and contains only even powers of $f$, in agreement with Lax *et al.* [1], who demonstrated that the transverse and longitudinal field components should be either even or odd, respectively. Note that when $l$ is odd, the contributions to the field are imaginary if $a_{n,l}$ is real and that the Laguerre polynomial is of a half-integer order. Half-integer Laguerre polynomials have known analytic forms in terms of modified Bessel functions of the first kind, $I_\nu(x)$. Note that the Gaussian term still enforces convergence for large $r$.

The remaining field components may be derived in the standard fashion by separating the source-free Maxwell equations into transverse and longitudinal components [13,14], following the discussion given by Cicchitelli *et al.* [2]. Considering the relative simplicity of the generalized field components above, it is important to consider the error introduced by neglecting the second integral in Eq. (9). Denoting the second, neglected, integral in Eq. (9) as $I_{\text{error}}(r)$ and noting that $f$ is small, the Laguerre polynomial can be approximated as its highest-order term $L_{c_-}^{(l)}(\frac{b^2}{2f^2}) \le (\frac{b^2}{2f^2})^{c_-} = (\frac{b}{\sqrt{2}f})^{n-l}$, because it is an alternating series. Furthermore, because the Bessel function of order 0 is bounded by unity for all real arguments, the inequality

$$I_{\text{error}}(r) \le \frac{1}{f^{n+2}} \int_1^\infty \left(\frac{b}{\sqrt{2}}\right)^{2m+n} \exp\left[-\frac{b^2}{2f^2}\right] b\mathrm{d}b \qquad (13)$$

holds. This integral can be evaluated analytically and taking the most extreme term with $n = N$ yields

$$\frac{1}{2^{m+N/2} f^{N+2}} \int_1^\infty \exp\left[-\frac{u}{2f^2}\right] u^{m+N/2} \mathrm{d}u$$
$$= \exp\left[-\frac{1}{2f^2}\right] \sum_{j=0}^{m+\frac{N}{2}} \frac{\left(m+\frac{N}{2}\right)!}{j!} \frac{1}{2^m f^2 (2f^2)^{\frac{N}{2}-j+1}}, \qquad (14)$$

when $N$ is even. Thus the error incurred in each term using the separation of Eq. (9) may be estimated for any values of $f$ and $N$ and adding 1 to an odd $N$. For reasonable values of $N \le 100$, $f \sim 0.03$ allows for accuracy greater than $10^{-10}$.

The preceding development generalizes the typical Gaussian source function in an aperture to include a more realistic and flexible radial and angular dependence. The field components describe flat-top profiles and annular beams, as well as Laguerre–Gaussian beams. It is clear that flattened Gaussian profiles are a form of the Laguerre–Gaussian profile without an angular momentum term. Furthermore, the effect of the nonparaxial terms is explicit in the structure of the field components and in the error estimates, facilitating numerical evaluations of the field. This approach thus represents a significant simplification of previous research.

Additionally, this approach is equally valid when looking at the expansion of the magnetic fields. Given a coherent, linearly polarized beam of light in the aperture with $E_y = 0$ and $B_x = 0$, there will be field components generated by both fields, as mentioned by Sepke and Umstadter [4]. These two sets of fields beyond the aperture must be added together to form the complete solution. Typically the terms from the magnetic field are neglected in the aperture, because they are an order in $f$ smaller for the electric field, which is the field of interest in many experiments. However, it is conceivable to imagine a system of magneto-optical tweezers, for example, which work similarly to ordinary optical tweezer systems but instead use the magnetic properties of the particle to trap it. For such an experiment one would be more interested in the analogous derivation, which starts with the magnetic field of the beam.

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