Acousto-optic finite-difference frequency-domain algorithm for first-principles simulations of on-chip acousto-optic devices

Supplementary Material

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Part 1. Explicit numerical formalism in two-dimensions

In Part 1 of the Supplementary Material, we present the details of the two-dimensional acousto-optic FDFD algorithm for the transverse-electric (TE) polarization, where the electric field E_z couples to the acoustic displacement fields U_x and U_y. This formalism is used for the simulation examples shown in Figs. 3 and 4.

To start, we recall the general first-principles equations for both optical and acoustic waves. For an optical wave, the electric field at frequency ω is described by

$$\nabla \times \mu_0^{-1} \nabla \times \mathbf{E}(\omega) - \omega^2 \varepsilon_0 \varepsilon_r \mathbf{E}(\omega) - \omega^2 \mathbf{P}(\omega) = -i\omega \mathbf{J}(\omega) , \quad \text{(S1.1)}$$

and for an acoustic wave, the mechanical displacement field at frequency Ω is described by

$$\rho \Omega^2 U_i + \sum_{jkl} \partial_j \left[c_{ijkl} + \eta_{ijkl} \frac{\partial}{\partial t} \right] \partial_k U_l + F_i = 0 . \quad \text{(S1.2)}$$

In two-dimensions, we adopt a discretization grid as shown in Fig. S1(a). In each *ij* cell, where *i* and *j* are the indices in the *x* and *y* directions respectively, E_z is located at the origin, $U_{x(y)}$ is located at the half point along the $x(y)$ direction, and the material parameters are specified at the center of the cell.

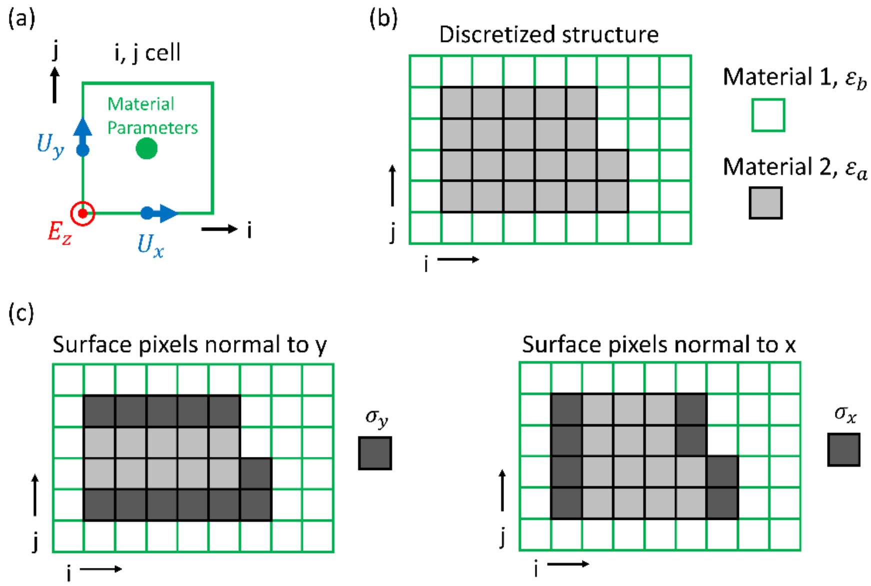

Figure S1. (a) Illustration of the placement of the fields and material parameters inside the discretized *ij* cell. (b) Schematics of an arbitrary structure. (c) Illustration of the surface-normal elements σ_y and σ_x of the arbitrary structure in (b).

To construct the finite difference equations of Eqns. (S1.1) and (S1.2) for E_z, U_x, and U_y, we first introduce a few operators. In the equations below, ∂_w^f and ∂_w^b denote a forward and backward derivative operator in direction w, respectively. Likewise, V_w^f and V_w^b denote a forward and backward averaging operator, respectively. For a given material parameter, a superscript with $< w >$ denotes that the parameter is being averaged in the w direction. This is to ensure that the parameters are co-located with the field components that they act on. With these notations, Eqn. (S1.1) for an electric field $E_{z,m}$ at frequency ω_m can be written as:

$$\left[\partial_x^b \mu_0^{-1} \partial_x^f + \partial_y^b \mu_0^{-1} \partial_y^f + \omega_m^2 \varepsilon_0 \varepsilon_r^{<z>}\right] E_{z,m} + \omega_m^2 P_{z,m} = i\omega_m J_{z,m}, \qquad \text{(S1.3)}$$

and Eqn. (S1.2) becomes:

$$\left[\Omega^2\rho^{<y>}+\partial_x^f c_{11}^{<z>}\partial_x^b+\partial_y^b c_{66}\partial_y^f\right]U_x+\left[\partial_x^f c_{12}^{<z>}\partial_y^b+\partial_y^b c_{66}\partial_x^f\right]U_y+F_x=0\,, \qquad \text{(S1.4a)}$$

$$\left[\partial_x^b c_{66}\partial_y^f+\partial_y^f c_{12}^{<z>}\partial_x^b\right]U_x+\left[\Omega^2\rho^{<x>}+\partial_x^b c_{66}\partial_x^f+\partial_y^f c_{11}^{<z>}\partial_y^b\right]U_y+F_y=0\,. \qquad \text{(S1.4b)}$$

In Eqns. (S1.3) and (S1.4), we assume that all the field, material quantities, and operators have been discretized, and the fields are reshaped into a column vector. For the acoustic parameters, we adopt the Voigt index notation such that in c_{IJ}, I and J are index from 1 to 6, where $1 = xx;\ 2 = yy;\ 3 = zz;\ 4 = yz = zy;\ 5 = xz = zx;\ 6 = xy = yx$. Furthermore, the terms c_{IJ} is a complex tensor that contains both the stiffness tensor and the viscosity tensor: $c_{IJ} \equiv c_{IJ} + i\Omega\eta_{IJ}$. To shorten the operators, Eqns. (S1.3) and (S1.4) can be written as:

$$A_m E_{z,m}+\omega_m^2 P_{z,m}=i\omega_m J_{z,m}\,, \qquad \text{(S1.5)}$$

$$B_{11}U_x+B_{12}U_y+F_x=0\,, \qquad \text{(S1.6a)}$$

$$B_{21}U_x+B_{22}U_y+F_y=0\,. \qquad \text{(S1.6b)}$$

To include the functional forms of the coupling between E_z, U_x, and U_y, we first identify the constituents of $P_{z,m}$, F_x, and F_y just like that of Eqns. (7) and (17):

$$P_{z,m}=P_{z,m}^{(b,PE)}+P_{z,m}^{(s,MB)}\,, \qquad \text{(S1.7)}$$

$$F_{x,y}=F_{x,y}^{(b,ES)}+F_{x,y}^{(s,ES)}+F_{x,y}^{(s,MB)}\,. \qquad \text{(S1.8)}$$

First, we expand the terms in Eqn. (S1.7). This term is the same as Eqn. (13) of the manuscript. With the polarization that we chose in our paper, the photo-elastic coupling tensor only happens through the $p_{12} \equiv p_{zzxx} = p_{zzyy}$ element of the tensor. Thus, we have:

$$P_{z,m}^{(b,PE)} = \varepsilon_0 \varepsilon_r^{\ 2}\ p_{12}^{<z>} \left[\left(\partial_x^b U_x + \partial_y^b U_y \right) E_{z,m-1} + \left(\partial_x^b U_x^* + \partial_y^b U_y^* \right) E_{z,m+1} \right]. \tag{S1.9}$$

For moving boundary considerations, since E_z is transverse to all material boundaries that lie in the $x - y$ plane, Eqn. (14) contains only one term, namely

$$\Delta D_z = \varepsilon_0 (\varepsilon_a - \varepsilon_b) E_z \,. \tag{S1.10}$$

With the boundary definition of $\boldsymbol{\sigma}^s$ (an example of which is shown in Fig. S1(c) for the structure in Fig. S1(b)), Eqn. (15) becomes:

$$\begin{aligned} P_{z,m}^{(s,MB)} &= \varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(V_x^b (\sigma_x^{<y>} U_x) + V_y^b (\sigma_y^{<x>} U_y) \right) E_{z,m-1} \\ &+ \varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(V_x^b (\sigma_x^{<y>} U_x^*) + V_y^b (\sigma_y^{<x>} U_y^*) \right) E_{z,m+1}. \end{aligned} \tag{S1.11}$$

Together, Eqns. (S1.3), (S1.9), and (S1.11) describe the optical equation of motion.

Now, we treat the acoustic wave equation and expand the terms in Eqn. (S1.8). In the x direction, the electrostrictive body and surface forces in Eqn. (18) can be written as:

$$F_x^{(b,ES)} = -\varepsilon_0 p_{12}^{<y>} (\varepsilon_r^{<y>})^2 \partial_x^f \sum_m E_{z,m}^* \, E_{z,m+1} \,, \tag{S1.12a}$$

$$F_x^{(s,ES)} = \varepsilon_0 p_{12}^{<y>} (\varepsilon_r^{<y>})^2 \sigma_x^{<y>} V_x^f \sum_m E_{z,m}^* \, E_{z,m+1} \,. \tag{S1.12b}$$

The radiation pressure term in Eqn. (20) can be written as:

$$F_x^{(s,MB)} = \varepsilon_0 (\varepsilon_a - \varepsilon_b) \sigma_x^{<y>} V_x^f \sum_m E_{z,m}^* \, E_{z,m+1} \,. \tag{S1.13}$$

Similarly, the forces in the y direction can be written as:

$$F_y^{(b,ES)} = -\varepsilon_0 p_{12}^{<x>} (\varepsilon_r^{<x>})^2 \partial_y^f \sum_m E_{z,m}^* \, E_{z,m+1} \,, \tag{S1.14a}$$

$$F_y^{(s,ES)} = \varepsilon_0 p_{12}^{<x>} (\varepsilon_r^{<x>})^2 \sigma_y^{<x>} V_y^f \sum_m E_{z,m}^* E_{z,m+1} , \tag{S1.14b}$$

$$F_y^{(s,MB)} = \varepsilon_0 (\varepsilon_a - \varepsilon_b) \sigma_y^{<x>} V_y^f \sum_m E_{z,m}^* E_{z,m+1} . \tag{S1.15}$$

By combining all the constituents of $P_{z,m}$, F_x, and F_y into Eqns. (S1.5) and (S1.6), we obtain the discretized dynamic equations as described in Eqns. (16) and (21). Explicitly, with the choice of polarization in two-dimensions, they can be written as:

$$\begin{aligned} A_m E_{z,m} &+ \omega_m^2 \varepsilon_0 \varepsilon_r{}^2 \, p_{12}^{<z>} \left[\left(\partial_x^b U_x + \partial_y^b U_y \right) E_{z,m-1} + \left(\partial_x^b U_x^* + \partial_y^b U_y^* \right) E_{z,m+1} \right] \\ &+ \omega_m^2 \varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(V_x^b (\sigma_x^{<y>} U_x) + V_y^b (\sigma_y^{<x>} U_y) \right) E_{z,m-1} \\ &+ \omega_m^2 \varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(V_x^b (\sigma_x^{<y>} U_x^*) + V_y^b (\sigma_y^{<x>} U_y^*) \right) E_{z,m+1} = i\omega_m J_{z,m}, \end{aligned} \tag{S1.16}$$

$$\begin{aligned} B_{11} U_x + B_{12} U_y &- \varepsilon_0 p_{12}^{<y>} (\varepsilon_r^{<y>})^2 \partial_x^f \sum_m E_{z,m}^* E_{z,m+1} \\ &+ \varepsilon_0 \left[p_{12}^{<y>} (\varepsilon_r^{<y>})^2 + (\varepsilon_a - \varepsilon_b) \right] \sigma_x^{<y>} V_x^f \sum_m E_{z,m}^* E_{z,m+1} = 0, \end{aligned} \tag{S1.17a}$$

$$\begin{aligned} B_{21} U_x + B_{22} U_y &- \varepsilon_0 p_{12}^{<x>} (\varepsilon_r^{<x>})^2 \partial_y^f \sum_m E_{z,m}^* E_{z,m+1} \\ &+ \varepsilon_0 \left[p_{12}^{<x>} (\varepsilon_r^{<x>})^2 + (\varepsilon_a - \varepsilon_b) \right] \sigma_y^{<x>} V_y^f \sum_m E_{z,m}^* E_{z,m+1} = 0. \end{aligned} \tag{S1.17b}$$

To solve Eqns. (S1.16) and (S1.17), we can follow the techniques discussed in Sec. III of the manuscript by constructing the nonlinear system of equations and its Jacobian, and apply Newton's update equations to iteratively obtain a self-consistent set of equations.

Part 2. The explicit expression of the Jacobian operator

In Part 2 of the Supplementary Material, we present the general expression of the Jacobian operator term by term. The results are used for the Newton update equations in Eqns. (28) and (29). Recall that Eqn. (27) states that

$$D_{\mathbf{g}}(\mathbf{v}) = \frac{\partial \mathbf{g}(\mathbf{v})}{\partial \mathbf{v}} = \hat{O} + \frac{\partial \mathbf{C}(\mathbf{v})}{\partial \mathbf{v}}\ , \tag{S2.1}$$

where $\hat{O}$ is given in Eqn. (24), and the constituents of $\partial\mathbf{C}/\partial\mathbf{v}$ are given in Eqn. (26). To derive the $\partial\mathbf{C}/\partial\mathbf{v}$ term of Eqn. (S2.1), we first define an all-ones vector as $\mathbf{1} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$. With this notation, and by setting $\mathbf{E}_0 = \mathbf{E}_{M+1} = 0$, the $\partial\mathbf{C}/\partial\mathbf{v}$ term can be explicitly written term-by-term as follows:

$$\begin{aligned}\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{m-1}} &\equiv \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{E}_{m-1}} \\ &= -\omega_m{}^2\varepsilon_0[\varepsilon_r{}^2\left(\underline{\mathbf{p}}:\nabla\otimes\mathbf{U}\right)\mathbf{1} + (\varepsilon_a-\varepsilon_b)\hat{n}\times(-\hat{n}\times\mathbf{1})\left(\mathbf{U}\bullet\boldsymbol{\sigma}^s\right) \\ &\qquad + (\varepsilon_b^{-1}-\varepsilon_a^{-1})\hat{n}\hat{n}\bullet\left(\varepsilon_r\mathbf{1}\right)\left(\mathbf{U}\bullet\boldsymbol{\sigma}^s\right)],\end{aligned} \tag{S2.2}$$

$$\begin{aligned}\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{m+1}} &\equiv \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{E}_{m+1}} \\ &= -\omega_m{}^2\varepsilon_0[\varepsilon_r{}^2\left(\underline{\mathbf{p}}:\nabla\otimes\mathbf{U}^*\right)\mathbf{1} + (\varepsilon_a-\varepsilon_b)\hat{n}\times(-\hat{n}\times\mathbf{1})\left(\mathbf{U}^*\bullet\boldsymbol{\sigma}^s\right) \\ &\qquad + (\varepsilon_b^{-1}-\varepsilon_a^{-1})\hat{n}\hat{n}\bullet\left(\varepsilon_r\mathbf{1}\right)\left(\mathbf{U}^*\bullet\boldsymbol{\sigma}^s\right)],\end{aligned} \tag{S2.3}$$

$$\begin{aligned}\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{2M+1}} &\equiv \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{U}} \\ &= -\omega_m{}^2\varepsilon_0\varepsilon_r{}^2\left(\underline{\mathbf{p}}:\nabla\otimes\mathbf{1}\right)\mathbf{E}_{m-1} - \omega_m{}^2\varepsilon_0(\varepsilon_a-\varepsilon_b)\hat{n}\times\left[-\hat{n}\times\mathbf{E}_{m-1}\left(\mathbf{1}\bullet\boldsymbol{\sigma}^s\right)\right] \\ &\quad -\omega_m{}^2\varepsilon_0(\varepsilon_b^{-1}-\varepsilon_a^{-1})\hat{n}\left[\hat{n}\bullet\left(\varepsilon_r\mathbf{E}_{m-1}\right)\left(\mathbf{1}\bullet\boldsymbol{\sigma}^s\right)\right],\end{aligned} \tag{S2.4}$$

$$\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{2M+2}} \equiv \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{U}^*}$$

$$= -\omega_m{}^2 \varepsilon_0 \varepsilon_r{}^2 \left[\left(\underline{\mathbf{p}} : \nabla \otimes \mathbf{1} \right) \mathbf{E}_{m+1} \right] - \omega_m{}^2 \varepsilon_0 (\varepsilon_a - \varepsilon_b) \hat{n} \times \left[-\hat{n} \times \mathbf{E}_{m+1} \left(\mathbf{1} \bullet \boldsymbol{\sigma}^s \right) \right] \tag{S2.5}$$

$$- \omega_m{}^2 \varepsilon_0 (\varepsilon_b^{-1} - \varepsilon_a^{-1}) \hat{n} \left[\hat{n} \bullet \left(\varepsilon_r \mathbf{E}_{m+1} \right) \left(\mathbf{1} \bullet \boldsymbol{\sigma}^s \right) \right]$$

$$\frac{\partial \mathbf{C}_{m+M}(\mathbf{v})}{\partial \mathbf{v}_{M+m-1}} \equiv \frac{\partial \mathbf{K}_m^*(\mathbf{v})}{\partial \mathbf{E}_{m-1}^*} = \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{E}_{m+1}} \equiv \frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{m+1}}, \tag{S2.6}$$

$$\frac{\partial \mathbf{C}_{m+M}(\mathbf{v})}{\partial \mathbf{v}_{M+m+1}} \equiv \frac{\partial \mathbf{K}_m^*(\mathbf{v})}{\partial \mathbf{E}_{m+1}^*} = \frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{E}_{m-1}} = \frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{m-1}}. \tag{S2.7}$$

$$\frac{\partial \mathbf{C}_{m+M}(\mathbf{v})}{\partial \mathbf{v}_{2M+1}} \equiv \frac{\partial \mathbf{K}_m^*(\mathbf{v})}{\partial \mathbf{U}} = \left(\frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{U}^*} \right)^* \equiv \left(\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{2M+2}} \right)^*, \tag{S2.8}$$

$$\frac{\partial \mathbf{C}_{m+M}(\mathbf{v})}{\partial \mathbf{v}_{2M+2}} \equiv \frac{\partial \mathbf{K}_m^*(\mathbf{v})}{\partial \mathbf{U}^*} = \left(\frac{\partial \mathbf{K}_m(\mathbf{v})}{\partial \mathbf{U}} \right)^* \equiv \left(\frac{\partial \mathbf{C}_m(\mathbf{v})}{\partial \mathbf{v}_{2M+1}} \right)^*. \tag{S2.9}$$

$$\frac{\partial \mathbf{C}_{2M+1}(\mathbf{v})}{\partial \mathbf{v}_m} = \frac{\partial \mathbf{L}(\mathbf{v})}{\partial \mathbf{E}_m}$$

$$= \varepsilon_0 \varepsilon_r^2 (\boldsymbol{\sigma}^b - \nabla) \bullet \underline{\mathbf{p}} : \mathbf{E}_{m-1}^* \otimes \mathbf{1} \tag{S2.10}$$

$$+ \boldsymbol{\sigma}^b \left[\varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(\hat{n} \times \mathbf{E}_{m-1}^* \right) \left(\hat{n} \times \mathbf{1} \right) - \varepsilon_0 (\varepsilon_b^{-1} - \varepsilon_a^{-1}) \left(\hat{n} \bullet \varepsilon_r \mathbf{E}_{m-1}^* \right) \left(\hat{n} \bullet \varepsilon_r \mathbf{1} \right) \right],$$

$$\frac{\partial \mathbf{C}_{2M+1}(\mathbf{v})}{\partial \mathbf{v}_{M+m}} = \frac{\partial \mathbf{L}(\mathbf{v})}{\partial \mathbf{E}_m^*}$$

$$= \varepsilon_0 \varepsilon_r^2 (\boldsymbol{\sigma}^b - \nabla) \bullet \underline{\mathbf{p}} : \mathbf{1} \otimes \mathbf{E}_{m+1} \tag{S2.11}$$

$$+ \boldsymbol{\sigma}^b \left[\varepsilon_0 (\varepsilon_a - \varepsilon_b) \left(\hat{n} \times \mathbf{1} \right) \left(\hat{n} \times \mathbf{E}_{m+1} \right) - \varepsilon_0 (\varepsilon_b^{-1} - \varepsilon_a^{-1}) \left(\hat{n} \bullet \varepsilon_r \mathbf{1} \right) \left(\hat{n} \bullet \varepsilon_r \mathbf{E}_{m+1} \right) \right];$$

$$\frac{\partial \mathbf{C}_{2M+2}(\mathbf{v})}{\partial \mathbf{v}_m} = \frac{\partial \mathbf{L}^*(\mathbf{v})}{\partial \mathbf{E}_m} = \left(\frac{\partial \mathbf{L}(\mathbf{v})}{\partial \mathbf{E}_m^*} \right)^* = \left(\frac{\partial \mathbf{C}_{2M+1}(\mathbf{v})}{\partial \mathbf{v}_{M+m}} \right)^*, \tag{S2.12}$$

$$\frac{\partial \mathbf{C}_{2M+1}(\mathbf{v})}{\partial \mathbf{v}_{M+m}} = \frac{\partial \mathbf{L}^*(\mathbf{v})}{\partial \mathbf{E}_m^*} = \left(\frac{\partial \mathbf{L}(\mathbf{v})}{\partial \mathbf{E}_m}\right)^* = \left(\frac{\partial \mathbf{C}_{2M+1}(\mathbf{v})}{\partial \mathbf{v}_m}\right)^* . \tag{S2.13}$$

With Eqns. (S2.2) to (S2.13), we can apply the Newton update equations in Eqn. (28) and (29).

Part 3. Coupled mode theory for an acousto-optic waveguide.

In Part 3 of the Supplementary Material, we derive the interaction of optical and acoustic waves inside an acousto-optic medium using coupled mode theory. This derivation is done also in the transverse electric polarization where E_z couples to U_x and U_y, and the results are used for the numerical validation in Fig. 3.

In the stimulated Brillouin scattering (SBS) process considered in the paper, there exists three interacting waves: a back-propagating optical pump $\tilde{E}_2(x,y,t)$ at ω_2 with propagation constant β_2, a forward-propagating optical Stokes wave $\tilde{E}_1(x,y,t)$ at ω_1 with propagation constant β_1, and a back-propagating acoustic wave $\tilde{\mathbf{U}}(x,y,t)$ at frequency $\Omega = \omega_2 - \omega_1$ and wave vector $q = \beta_2 - \beta_1$. Mathematically, these fields can be written as [1]:

$$\tilde{E}_1(x,y,t) \equiv E_1(x,y,t)e^{i\omega_1 t} + c.c. = A_1(x)\hat{e}_1(y)e^{-i\beta_1 x + i\omega_1 t} + c.c.\,, \tag{S3.1}$$

$$\tilde{E}_2(x,y,t) \equiv E_2(x,y,t)e^{i\omega_2 t} + c.c. = A_2(x)\hat{e}_2(y)e^{-i\beta_2 x + i\omega_2 t} + c.c.\,, \tag{S3.2}$$

$$\tilde{\mathbf{U}}(x,y,t) \equiv \mathbf{U}(x,y,t)e^{i\Omega t} + c.c. = B(x)\left[\hat{x}\hat{u}_x(y) + \hat{y}\hat{u}_y(y)\right] + c.c.\,, \tag{S3.3}$$

where $A_{1,2}(x)$ and $B(x)$ are the slowly-varying modal amplitude for the optical waves and acoustic wave, respectively, and $\hat{e}_{1,2}(y)$ and $\hat{u}_{x,y}(y)$ are the modal profiles that satisfy the following optical and acoustic waveguide dispersion [1-3]:

$$(\partial_y^2 - \beta_m^2)\hat{e}_m(y) = -\omega_n^2 \mu_0 \varepsilon_0 \varepsilon_r \hat{e}_m(y)\,, \tag{S3.4}$$

$$\begin{pmatrix} q^2 c_{11} - \partial_y c_{66} \partial_y & iqc_{12}\partial_y + iq\partial_y c_{66} \\ iq\partial_y c_{12} + iqc_{66}\partial_y & q^2 c_{66} - \partial_y c_{11} \partial_y \end{pmatrix} \begin{pmatrix} \hat{u}_x \\ \hat{u}_y \end{pmatrix} = \rho \Omega_B{}^2 \begin{pmatrix} \hat{u}_x \\ \hat{u}_y \end{pmatrix}, \tag{S3.5}$$

where ε_r is the relative permittivity of the medium, c_{IJ} describes the stiffness tensor in Voigt notation, ρ is the density of the acoustic material, and Ω_B is the SBS frequency of the waveguide with the given wave vector $q = \beta_2 - \beta_1$, which may be detuned from the acoustic frequency $\Omega = \omega_2 - \omega_1$. Furthermore, the value of $\left|A_{1,2}(x)\right|^2$ and $|B(x)|^2$ are the optical and acoustic power of the guided mode, respectively. By this formulation, the optical modal profiles $\hat{e}_m(y)$ are normalized in the following way [3]:

$$\int \hat{e}_m{}^*(y)\hat{e}_m(y)dy = \frac{\omega_m \mu_0}{2\beta_m}, \tag{S3.6}$$

and likewise, $\hat{\mathbf{u}}(y)$ is normalized as [1]

$$P_b = 2i\Omega \int dydz \sum_{ikl} c_{xikl} \hat{u}_i{}^* \partial_k \hat{u}_l = 1, \tag{S3.7}$$

or more explicitly in two-dimensions,

$$\int dydz \left[-c_{11} iq \hat{u}_x^* \hat{u}_x + c_{12} \hat{u}_x^* \partial_y \hat{u}_y - c_{66} iq \hat{u}_y^* \hat{u}_y + c_{66} \hat{u}_y^* \partial_y \hat{u}_x \right] = \frac{1}{2i\Omega}. \tag{S3.8}$$

We now consider the acousto-optic interaction and derive the equation of motion for the envelope functions. For the optical waves in Eqns. (S3.1) and (S3.2), the equation of motion from Maxwell's equations is described by

$$\nabla^2 \tilde{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \tilde{E}}{\partial t^2} + \mu_0 \varepsilon_0 \frac{\partial^2 \chi^{(pe)} \tilde{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \Delta \tilde{D}}{\partial t^2}. \tag{S3.9}$$

To begin, we consider the optical envelope functions. By substituting Eqn. (S3.1) into Eqn. (S3.9) and applying the slowly-varying envelope approximation (SVEA) to ignore the $\partial_x^2 A_m$ terms [3], and together with the modal equation in Eqn. (S3.4), we obtain

$$2i\beta_1\hat{e}_1\partial_x A_1 = \omega_1{}^2\mu_0\varepsilon_0\left[\chi^{(b,PE)}E\right]_{\omega_1} + \omega_1{}^2\mu_0\left[\Delta D^{(s,MB)}\right]_{\omega_1}, \tag{S3.10}$$

and likewise for Eqn. (S3.2), we get

$$2i\beta_2\hat{e}_2\partial_x A_2 = \omega_2{}^2\mu_0\varepsilon_0\left[\chi^{(b,PE)}E\right]_{\omega_2} + \omega_2{}^2\mu_0\left[\Delta D^{(s,MB)}\right]_{\omega_2}. \tag{S3.11}$$

In Eqns. (S3.10) and (S3.11), the $[\ \cdot\ ]_{\omega_m}$ notation denotes summing up all terms with frequency ω_m. Upon applying the projection operation $\int dydz\ \hat{e}_1^*(\cdot)$ to Eqn. (S3.10) and $\int dydz\ \hat{e}_2^*(\cdot)$ to Eqn. (S3.11), and using the normalization conditions in Eqn. (S3.6), we get

$$\partial_x A_1 = -i\omega_1\varepsilon_0\int dydz\left\{\hat{e}_1^*\left[\chi^{(b,PE)}E\right]_{\omega_1}\right\} - i\omega_1\int dydz\left\{\hat{e}_1^*\left[\Delta D^{(s,MB)}\right]_{\omega_1}\right\}, \tag{S3.12}$$

$$\partial_x A_2 = -i\omega_2\varepsilon_0\int dydz\left\{\hat{e}_2^*\left[\chi^{(b,PE)}E\right]_{\omega_2}\right\} - i\omega_2\int dydz\left\{\hat{e}_2^*\left[\Delta D^{(s,MB)}\right]_{\omega_2}\right\}. \tag{S3.13}$$

In a similar fashion, we can derive the equation of motion for the acoustic wave. For the acoustic wave in Eqn. (S3.3), the equation of motion is described by

$$\rho\frac{\partial^2\tilde{U}_x}{\partial t^2} - \partial_x c'_{11}\,\partial_x\tilde{U}_x - \partial_x c'_{12}\,\partial_y\tilde{U}_y - \partial_y c'_{66}\,\partial_x\tilde{U}_y - \partial_y c'_{66}\,\partial_y\tilde{U}_x = F_x, \tag{S3.14}$$

$$\rho\frac{\partial^2\tilde{U}_y}{\partial t^2} - \partial_y c'_{11}\,\partial_y\tilde{U}_y - \partial_y c'_{12}\,\partial_x\tilde{U}_x - \partial_x c'_{66}\,\partial_x\tilde{U}_y - \partial_x c'_{66}\,\partial_y\tilde{U}_x = F_y, \tag{S3.15}$$

where c'_{IJ} is the complex-valued tensor that contains both the stiffness tensor c_{IJ} and the viscosity tensor η_{IJ} like so: $c'_{IJ} = c_{IJ} + i\Omega\eta_{IJ}$. As we substitute Eqn. (S3.3) into Eqns. (S3.14) and (S3.15) and use the SVEA by ignoring $\partial_x^2 B$ terms, the resulting equations of motion in the x and y directions respectively are:

$$\begin{aligned}&\left[-2iqc_{11}\hat{u}_x + c_{12}\partial_y\hat{u}_y + \partial_y c_{66}\hat{u}_y\right]\partial_x B\\ &\quad -\left[q^2\eta_{11} + iq\eta_{12}\partial_y\hat{u}_y + iq\partial_y\eta_{66}\hat{u}_y - \partial_y\eta_{66}\partial_y\hat{u}_x\right]i\Omega B = -F_x,\end{aligned} \tag{S3.16}$$

$$\begin{aligned}&\left[-2iqc_{66}\hat{u}_y + c_{66}\partial_y\hat{u}_x + \partial_y c_{12}\hat{u}_x\right]\partial_x B\\ &\quad -\left[q^2\eta_{66} + iq\eta_{66}\partial_y\hat{u}_x + iq\partial_y\eta_{12}\hat{u}_x - \partial_y\eta_{11}\partial_y\hat{u}_y\right]i\Omega B = -F_y.\end{aligned} \tag{S3.17}$$

When we project $\int dydz\,(\hat{x}\,\hat{u}_x^* + \hat{y}\,\hat{u}_y^*)(\,\cdot\,)$ onto Eqns. (S3.16) and (S3.17), and use the power normalization equation in Eqn. (S3.7), we find that

$$\partial_x B - \alpha B = i\Omega\int dydz\left[\hat{u}_x^* F_x + \hat{u}_y^* F_y\right], \tag{S3.18}$$

where α is the acoustic loss rate and has the form:

$$\alpha = \Omega^2\int dydz\left[\hat{u}_x^* L_x + \hat{u}_y^* L_y\right], \tag{S3.19}$$

$$L_x = -q^2\eta_{11}\,\hat{u}_x - iq\eta_{12}\,\partial_y\hat{u}_y - iq\partial_y\,\eta_{66}\hat{u}_y + \partial_y\eta_{66}\partial_y\hat{u}_x, \tag{S3.20}$$

$$L_y = \partial_y\eta_{11}\partial_y\hat{u}_y - iq\partial_y\eta_{12}\,\hat{u}_x - q^2\eta_{66}\,\hat{u}_y - iq\partial_y\,\eta_{12}\hat{u}_x. \tag{S3.21}$$

We are now ready to treat the acousto-optic coupling terms in the modal overlap intergrals of Eqns. (S3.12), (S3.13), and (S3.18). For the photo-elasticity term, with our choice of polarization,

$$\chi_{zz}^{(b,PE)} = {\varepsilon_r}^2 p_{12}\left(\partial_x U_x + \partial_y U_y\right). \tag{S3.22}$$

When applying this term to the electric field and matching frequency components, we explicitly evaluate the integrals with the photo-elastic terms to be

$$\begin{aligned}\varepsilon_0\int dydz\,\hat{e}_1^*\left[\chi^{(b,PE)}E\right]_{\omega_1} &= \varepsilon_0\int dydz\,\varepsilon_r{}^2 p_{12}\hat{e}_1^*\left(-iq\hat{u}_x+\partial_y\hat{u}_y\right)^*\hat{e}_2\,B^*A_2 \\ &\equiv Q_1{}^{(b,PE)}B^*A_2\end{aligned}, \tag{S3.23}$$

$$\begin{aligned}\varepsilon_0\int dydz\,\hat{e}_2^*\left[\chi^{(b,PE)}E\right]_{\omega_2} &= \varepsilon_0\int dydz\,\varepsilon_r{}^2 p_{12}\hat{e}_2^*\left(-iq\hat{u}_x+\partial_y\hat{u}_y\right)\hat{e}_1\,BA_1 \\ &\equiv Q_2{}^{(b,PE)}BA_1.\end{aligned}. \tag{S3.24}$$

To treat the boundary term, we recall from Part II of this document that [1]

$$\Delta D_z = \sigma_y^s\varepsilon_0(\varepsilon_a-\varepsilon_b)(\hat{y}\bullet\mathbf{U})E_z\,. \tag{S3.25}$$

After matching with the correct frequency terms, we can express Eqn. (S3.25) explicitly as

$$\left[\Delta D^{(s,MB)}\right]_{\omega_1} = \sigma_y^s\varepsilon_0(\varepsilon_a-\varepsilon_b)\hat{u}_y^*\hat{e}_2\,B^*A_2\,, \tag{S3.26}$$

$$\left[\Delta D^{(s,MB)}\right]_{\omega_2} = \sigma_y^s\varepsilon_0(\varepsilon_a-\varepsilon_b)\hat{u}_y\hat{e}_1\,BA_1\,. \tag{S3.27}$$

The integral with the moving boundary terms in Eqns. (S3.12) and (S3.13) become

$$\begin{aligned}\int dydz\,\hat{e}_1^*\left[\Delta D^{(s,MB)}\right]_{\omega_1} &= 2\left[\int dz\,(\varepsilon_a-\varepsilon_b)\,\hat{e}_1^*\hat{u}_y^*\hat{e}_2\right]_{y=a}B^*A_2 \\ &\equiv Q_1{}^{(s,MB)}B^*A_2,\end{aligned} \tag{S3.28}$$

$$\begin{aligned}\int dydz\,\hat{e}_2^*\left[\Delta D^{(s,MB)}\right]_{\omega_2} &= 2\left[\int dz\,(\varepsilon_a-\varepsilon_b)\,\hat{e}_2^*\hat{u}_y\hat{e}_1\right]_{y=a}BA_1 \\ &\equiv Q_2{}^{(s,MB)}BA_1,\end{aligned} \tag{S3.29}$$

where we choose $y = a$ to be the upper boundary of a symmetric waveguide with half-width a. In all of the coupling coefficient Q, we note $Q_2{}^{(b,PE)} = \left[Q_1{}^{(b,PE)}\right]^*$ and $Q_2{}^{(s,MB)} = \left[Q_1{}^{(s,MB)}\right]^*$, which is consistent with the results in Ref. 1. To summarize, the optical envelope functions vary with the following differential equations:

$$\partial_x A_1 = -i\omega_1\varepsilon_0 \left[Q_1^{(b,PE)} + Q_1^{(s,MB)} \right] B^* A_2 , \tag{S3.30}$$

$$\partial_x A_2 = -i\omega_2\varepsilon_0 \left[Q_2^{(b,PE)} + Q_2^{(s,MB)} \right] B A_1 . \tag{S3.31}$$

In the same way, we can expand the modal projection terms on the right hand side of Eqn. (S3.18) to see how the acoustic envelope couples to the electric fields. Just like before, the coupling can be decomposed into the electrostrictive contribution and the moving boundary contribution from radiation pressure. Analytically, we can rewrite (S3.18) as:

$$\partial_x B - \alpha B = i\Omega \left[Q_b^{(ES)} + Q_b^{(s,MB)} \right] A_1^* A_2 . \tag{S3.32}$$

As is stated in Ref. 1 and is independently verified, $Q_b^{(ES)} = Q_1^{(b,PE)}$ and $Q_b^{(s,MB)} = Q_1^{(s,MB)}$.

The coupled system of first-order differential equations (S3.30) to (S3.32) completely describes the equation of motion for the envelope functions $A_1(x)$, $A_2(x)$, and $B(x)$. The boundary conditions of $A_1(x)$, $A_2(x)$, and $B(x)$ are given *a priori* according to the setup of the SBS process. $|A_1(x)|^2$ and $|B(x)|^2$ give the power of the Stokes wave and acoustic wave, respectively, as plotted in Fig. 3(d).

References:

[1] C. Wolff, M. J. Steel, B. J. Eggleton, and C. G. Poulton. "Stimulated Brillouin scattering in integrated photonic waveguides: Forces, scattering mechanisms, and coupled-mode analysis," *Physical Review A* **92**, 013836 (2015).

[2] B. A. Auld. *Acoustic Fields and Waves in Solids,* 2nd edition (Krieger, 1990).

[3] H. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, 1984).