Supplementary information for: Achieving arbitrary control over pairs of polarization states using complex birefringent meta-materials

Alexander Cerjan and Shanhui Fan
Department of Electrical Engineering, and Ginzton Laboratory,
Stanford University, Stanford, California 94305, USA
(Dated: May 16, 2017)

I. POLARIZATION DYNAMICS OF THE ELECTROMAGNETIC FIELD AT THE EXCEPTIONAL POINT

In this section, we will derive the polarization dynamics of the electromagnetic field traveling along the z-axis of a complex birefringent meta-metamaterial when the system is exhibiting an exceptional point at $|\tau| = 1$. The result of this section is given in Eq. (15) of the main text for waves propagating in the $+z$ direction. At an exceptional point, the two eigenstates corresponding to forward propagation coallesce into a single, self-orthogonal eigenstate, and the matrix describing the system, G, has a non-trivial Jordan normal form. As such, the polarization dynamics must be expressed in terms of both the single remaining eigenstate, $|E^R\rangle$,

$$G|E_j^R\rangle = \lambda_j |E_j^R\rangle. \tag{S1}$$

as well as its associated Jordan vector, $|J^R\rangle$, which satisfies,

$$G|J_j^R\rangle = \lambda_j |J_j^R\rangle + |E_j^R\rangle. \tag{S2}$$

To facilitate our solution for the polarization dynamics when $|\tau| = 1$, it is then convenient to adopt a dimensionless system matrix G so that λ_j, $|E^R\rangle$ and $|J^R\rangle$ are also all dimensionless.

We begin with Maxwell's equations for a monochromatic signal,

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}, \tag{S3}$$

$$\nabla \times \mathbf{H} = -i\omega\bar{\varepsilon}\mathbf{E}, \tag{S4}$$

where the dielectric tensor is given by Eq. (6) in the main text. For a electromagnetic wave propagating in the z-direction, Maxwell's equations can be rewritten as

$$\left(\frac{1}{i\omega\sqrt{\varepsilon_r\mu}}\right)\frac{d}{dz}E_x = \sqrt{\frac{\mu}{\varepsilon_r}}H_y, \tag{S5}$$

$$\left(\frac{1}{i\omega\sqrt{\varepsilon_r\mu}}\right)\frac{d}{dz}E_y = \sqrt{\frac{\mu}{\varepsilon_r}}H_x, \tag{S6}$$

$$\left(\frac{1}{i\omega\sqrt{\varepsilon_r\mu}}\right)\left(\sqrt{\frac{\mu}{\varepsilon_r}}\right)\frac{d}{dz}H_x = -\eta E_x - (1+i\nu)E_y, \tag{S7}$$

$$\left(\frac{1}{i\omega\sqrt{\varepsilon_r\mu}}\right)\left(\sqrt{\frac{\mu}{\varepsilon_r}}\right)\frac{d}{dz}H_y = (1-i\nu)E_x + \eta E_y, \tag{S8}$$

in which $\eta = \varepsilon_a/\varepsilon_r$ and $\nu = \varepsilon_i/\varepsilon_r$. Now, by defining $F_j = \sqrt{\mu/\varepsilon_r}H_j$ and $\tilde{z} = i\omega\sqrt{\varepsilon_r\mu}z$, we can write down a dimensionless form of the evolution of the electromagnetic field as,

$$\frac{d}{d\tilde{z}}\begin{pmatrix} E_x \\ E_y \\ F_x \\ F_y \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ -\eta & -(1+i\nu) & & \\ (1-i\nu) & \eta & & \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ F_x \\ F_y \end{pmatrix} \equiv G \begin{pmatrix} E_x \\ E_y \\ F_x \\ F_y \end{pmatrix}. \tag{S9}$$

When $|\tau| \neq 1$, this equation reproduces the polarization dynamics of the electromagnetic field in Eq. (13) of the main text, with the same eigenvalues and eigenvectors given in Eqs. (10)-(12) of the main text for both forwards and backwards propagating fields, although the form given here also contains information about the magnetic fields of the eigenstates.

However, when $|\tau| = 1$, G has a non-trivial Jordan normal form, as the pairs of eigenvectors corresponding to the forwards and backwards traveling waves coalesce into a single eigenvector for each direction. Thus, we re-diagonalize

G by choosing a basis consisting of the remaining eigenstates as well as their associated Jordan vectors, given in Eq. (S2). Note that Eq. (S2) does not uniquely determine the associated Jordan vectors, as any multiple of the original eigenvector can still be added to $|J_j^R\rangle$. Typically, one would attempt to choose the Jordan vector to be orthogonal to its associated eigenvector for convenience. However, this is not possible at an exceptional point because the coalesced eigenvectors are already self-orthogonal. Thus, to uniquely determine the Jordan vectors here, we pick them to also be self-orthogonal, $\langle J_j^L|J_j^R\rangle = 0$, where the left Jordan vector is $\langle J_j^L| = (|J_j^R\rangle)^T$. With this, we can now place G in Jordan normal form as $J = P^{-1}GP$, where

$$J = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}, \tag{S10}$$

and

$$P = \left(|E_f^R\rangle, |J_f^R\rangle, |E_b^R\rangle, |J_b^R\rangle\right) = \begin{pmatrix} 1 & \left(\frac{i}{\eta} - \frac{1}{2}\right) & 1 & \left(-\frac{i}{\eta} + \frac{1}{2}\right) \\ i & \left(\frac{1}{\eta} - \frac{i}{2}\right) & i & \left(-\frac{1}{\eta} + \frac{i}{2}\right) \\ -i & \left(-\frac{1}{\eta} - \frac{i}{2}\right) & i & \left(-\frac{1}{\eta} - \frac{i}{2}\right) \\ 1 & \left(\frac{i}{\eta} + \frac{1}{2}\right) & -1 & \left(\frac{i}{\eta} + \frac{1}{2}\right) \end{pmatrix}. \tag{S11}$$

Thus, the polarization dynamics of an arbitrary field propagating along the z-axis through a complex birefringent material is given by

$$|E(z)\rangle = e^{ik_0 z}\left[\left(A_{E,f} + ik_0 z A_{J,f}\right)|E_f^R\rangle + A_{J,f}|J_f^R\rangle\right] + e^{-ik_0 z}\left[\left(A_{E,b} - ik_0 z A_{J,b}\right)|E_b^R\rangle + A_{J,b}|J_b^R\rangle\right], \tag{S12}$$

in which $k_0 = \omega\sqrt{\varepsilon_r \mu}$, and where the coefficients A can be found by calculating $P^{-1}|E(0)\rangle$.

II. GENERALIZED UNITARITY RELATION IN COMPLEX BIREFRINGENT SYSTEMS

Despite the fact that intensity is not conserved in complex birefringent materials, there are two generalized unitarity relations [1] that can be derived from considering the scattering matrix of the system. In the one-dimensional systems considered here, there are two input channels and two output channels at each edge of the device, corresponding to the incoming and outgoing waves with electric fields in the x- and y-directions. Thus, the scattering matrix gives the output field amplitudes, ϕ, in terms of the input field amplitudes, ψ, as

$$\begin{pmatrix} \phi_x^L \\ \phi_y^L \\ \phi_x^R \\ \phi_y^R \end{pmatrix} = S \begin{pmatrix} \psi_x^L \\ \psi_y^L \\ \psi_x^R \\ \psi_y^R \end{pmatrix}. \tag{S13}$$

As mentioned in the main text, complex birefringent materials are invariant upon permutation of the x and y axes, $\mathcal{M}$, and time reversal, $\mathcal{T}$, which both reverses the outputs and inputs of the system, and changes the gain to loss. Specifically,

$$\mathcal{M}\begin{pmatrix} \psi_x^L \\ \psi_y^L \\ \psi_x^R \\ \psi_y^R \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_x^L \\ \psi_y^L \\ \psi_x^R \\ \psi_y^R \end{pmatrix} = \begin{pmatrix} \psi_y^L \\ \psi_x^L \\ \psi_y^R \\ \psi_x^R \end{pmatrix}, \tag{S14}$$

and

$$\mathcal{T}\begin{pmatrix} \psi_x^L \\ \psi_y^L \\ \psi_x^R \\ \psi_y^R \end{pmatrix} = \begin{pmatrix} \phi_x^L \\ \phi_y^L \\ \phi_x^R \\ \phi_y^R \end{pmatrix}, \tag{S15}$$

assuming that the operating frequency of the system is real. Thus, as $\mathcal{MT}\psi = \phi$, and noting that $(\mathcal{MT})^2 = 1$, one can prove Eq. (17) in the main text,

$$(\mathcal{MT})S(\mathcal{MT}) = S^{-1}. \tag{S16}$$

To use this equation to derive constraints on the transmission and reflection coefficients of complex birefringent materials, we write the scattering matrix in block form [1],

$$S = \begin{pmatrix} \mathbf{r}_L & \mathbf{t} \\ \mathbf{t}^T & \mathbf{r}_R \end{pmatrix}, \tag{S17}$$

in which we have invoked reciprocity, $S = S^T$. Thus, Eq. (S16) can be rewritten as

$$\begin{pmatrix} M\mathbf{r}_L M & M\mathbf{t}M \\ M\mathbf{t}^T M & M\mathbf{r}_R M \end{pmatrix}^* = \begin{pmatrix} \mathbf{r}_L & \mathbf{t} \\ \mathbf{t}^T & \mathbf{r}_R \end{pmatrix}^{-1}, \tag{S18}$$

in which

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{S19}$$

by performing the matrix inversion in block form, one can find the four equations

$$M\mathbf{r}_L^* M = \left(\mathbf{r}_L - \mathbf{t}\mathbf{r}_R^{-1}\mathbf{t}^T\right)^{-1}, \tag{S20}$$

$$M\mathbf{r}_R^* M = \left(\mathbf{r}_R - \mathbf{t}^T\mathbf{r}_L^{-1}\mathbf{t}\right)^{-1}, \tag{S21}$$

$$\mathbf{r}_L M\mathbf{t}^* M = \mathbf{t}\left(\mathbf{r}_R - \mathbf{t}^T\mathbf{r}_L^{-1}\mathbf{t}\right)^{-1}, \tag{S22}$$

$$\mathbf{r}_R M\mathbf{t}^\dagger M = \mathbf{t}^T\left(\mathbf{r}_L - \mathbf{t}\mathbf{r}_R^{-1}\mathbf{t}^T\right)^{-1}. \tag{S23}$$

Finally, by combining Eqs. (S21) and (S22), as well as Eqs. (S20) and (S23), one finds the generalized unitarity relations

$$\mathbf{t}^T M\mathbf{t}^* - M = -\mathbf{r}_R M\mathbf{r}_R^*, \tag{S24}$$

$$\mathbf{t}M\mathbf{t}^\dagger - M = -\mathbf{r}_L M\mathbf{r}_L^*. \tag{S25}$$

III. DETERMINATION OF A COMPLEX BIREFRINGENT META-MATERIAL STRUCTURE

In this section, we first derive the dielectric tensor for the complex birefringent meta-material shown in Fig. 3 of the main text where both gain and loss are provided by semiconductor media. We then discuss a second example of such a meta-material which uses metal to provide loss, rather than a semiconductor medium. As discussed in the main text, the meta-material shown in Fig. 3 is comprised of interwoven layers of calcite and a semiconductor comb of two different materials which have gain and loss respectively. By arranging the semiconductor gain and loss in such a comb, the electric field component which is parallel to the arrangement of the layers sees a different effective dielectric than the field component which is perpendicular to these layers, so long as the layers are much smaller than the wavelength. Thus, the effective dielectric tensor for this comb is

$$\bar{\varepsilon}_{comb} = \begin{pmatrix} \varepsilon_\perp & 0 \\ 0 & \varepsilon_\parallel \end{pmatrix}, \tag{S26}$$

in which

$$\varepsilon_\parallel = x\varepsilon_{gain} + (1-x)\varepsilon_{loss}, \tag{S27}$$

$$\varepsilon_\perp = \frac{\varepsilon_{gain}\varepsilon_{loss}}{x\varepsilon_{loss} + (1-x)\varepsilon_{gain}}, \tag{S28}$$

and x is the width of the gain portion of the comb. Using the parameters specified in the main text, by setting $x = 0.7659$, $\varepsilon_{gain} = 4 - 0.1i$, and $\varepsilon_{loss} = 9.7406 + 0.6262i$, the dielectric tensor for the comb structure is

$$\bar{\varepsilon}_{comb} = \begin{pmatrix} 4.6438 - 0.0700i & 0 \\ 0 & 5.3439 + 0.0700i \end{pmatrix}. \tag{S29}$$

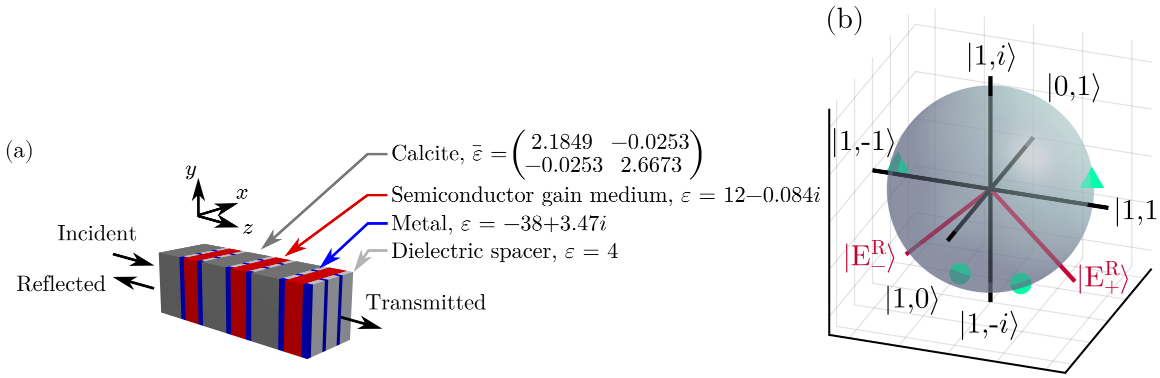

FIG. S1. (a) Schematic of a complex birefringent meta-material, consisting of layers of a conventional birefringent material (dark gray), a semiconductor gain medium (red), and a comb consisting of metal (blue) and a passive dielectric spacer (light gray). The patterning of the comb structure is assumed to be fine enough relative to the wavelength of the light so as to be in the effective medium limit. (b) Transformation of the polarization on the Poincaré sphere of two signals through $127\mu m$ of a complex birefringent meta-material, as shown schematically in (a). Here, for an incident light with wavelength $1.55\mu m$, we have used calcium carbonate whose ordinary and extraordinary axes are rotated 93° with respect to the x-axis. The remaining material dielectrics are $\varepsilon_{gain} = 12 - 0.064i$, $\varepsilon_{metal} = -38 + 3.47i$, and $\varepsilon_{spacer} = 4$. The comb layer consists of 95.2% dielectric spacer and 4.8% metal, and the effective dielectric tensor of the comb layer has $\varepsilon_{\parallel} = 1.9840 + 0.1666i$ and $\varepsilon_{\perp} = 4.2239 + 0.0020i$. The simulated structure is comprised of layers of calcite, with width $z_{cal} = 29.7nm$, and the semiconductor gain, with width $z_{gain} = 6.4nm$, separated by layers of the metal/dielectric comb structure with width $z_{comb} = 3.2nm$. These parameters yield a complex birefringent meta-material with a total effective medium of $\varepsilon_{xx} = 3.9630 - 0.0123i$, $\varepsilon_{yy} = 3.9647 + 0.0124i$, and $\varepsilon_{xy} = -0.0177$, which corresponds to $\tau = -0.69$. The initial signal polarizations are separated by 18° (cyan circles), while the final polarization states are nearly orthogonal (cyan triangles). The surrounding medium is air, $\varepsilon = 1$.

For $\lambda = 1.55\mu m$, the anisotropic tensor for calcite is

$$\bar{\varepsilon}_{cal}(\theta) = \begin{pmatrix} 2.4261 & 0 \\ 0 & 2.4261 \end{pmatrix} + 0.2425 \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \tag{S30}$$

where θ is the angle between the ordinary axis and the x-axis. As the light propagates along the z-axis through the structure, the total effective dielectric tensor is given by

$$\bar{\varepsilon}_T = f\bar{\varepsilon}_{cal} + (1-f)\bar{\varepsilon}_{comb}. \tag{S31}$$

for $30nm$ layers of calcite rotated 7.4657° interwoven with $20nm$ layers of dielectric gain/loss comb, $f = 0.6$, and

$$\bar{\varepsilon}_T = \begin{pmatrix} 3.4538 - 0.0280i & 0.0375 \\ 0.0375 & 3.4526 + 0.0280i \end{pmatrix}. \tag{S32}$$

Note that in the calculation shown in Fig. 3, the effective medium approximation along the z-axis was not used, and each of the alternating layers of calcite and dielectric comb were simulated directly.

The dielectric comb consisting of layers of gain and loss shown in Fig. 3 of the main text might present experimental challenges, such as carrier recombination at the edges of the gain medium. However, similar meta-materials can be designed which overcome this challenge, such as the complex birefringent meta-material shown in Fig. S1, wherein a solid dielectric gain slab is used to provide uniform gain, and then a comb structure of metal wires and dielectric spacers is used to provide anisotropic loss. Moreover, these wires could be used to supply current to the gain medium.

IV. DESIGN FOR A TUNABLE COMPLEX BIREFRINGENT META-MATERIAL

In Fig. 3 of the main text, we introduce a meta-material which is able to transform a particular pair of input states, $|\pm 16.2°\rangle$ to orthogonally polarized output states. One can turn this system into a tunable complex birefringent meta-material, such that many different choices of input states can still be made orthogonal without re-fabricating the systemx. First, we note that in this system, both the gain and loss are supplied by semiconductor media, and thus the amount of gain and loss provided by the gain/loss comb layer can be changed electrically. In particular, it is possible to change both the gain and the loss regions to be at their transparency point, where $\varepsilon_i = 0$ and thus $\tau = 0$,

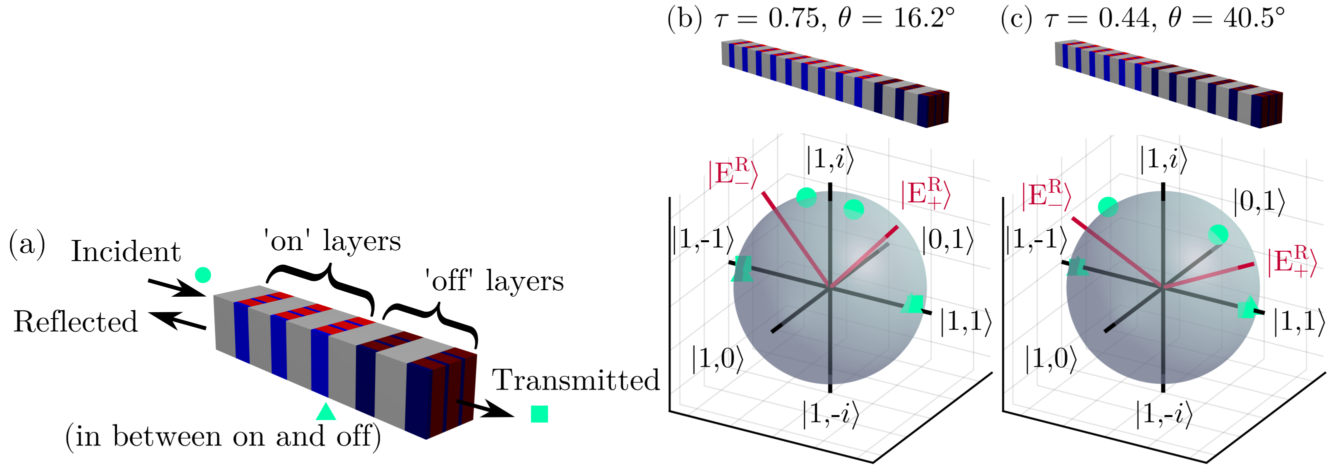

FIG. S2. Schematic of a tunable complex birefringent meta-material, consisting of layers of a conventional birefringent material (gray), and layers of a material containing gain (red) and loss (blue), forming a comb. Those unit cells in which the gain and loss are 'on' are shown in brighter colors relative to those unit cells which are 'off.' The patterning of the structure is assumed to be fine enough relative to the wavelength of the light so as to be in the effective medium limit. (b) Transformation of the polarization on the Poincaré sphere of two signals through $75\mu m$ of a tunable complex birefringent meta-material, of which the first $59\mu m$ are 'on' unit cells and the remaining $16\mu m$ are 'off' unit cells, as shown schematically in (a). Here, for an incident light with wavelength $1.55\mu m$, we have used calcium carbonate, whose ordinary and extraordinary axes are rotated $7.47°$ with respect to the lab frame, yielding a dielectric tensor with $\varepsilon_{xx} = 2.66$, $\varepsilon_{yy} = 2.19$, $\varepsilon_{xy} = 0.063$. The isotropic gain has $\varepsilon = 4 - 0.1i$, and the isotropic loss has $\varepsilon = 9.74 + 0.63i$. By forming a comb consisting of 77% gain regions and 23% loss regions, the effective dielectric tensor is anisotropic in this layer, with $\varepsilon_{\parallel} = 5.34 + 0.07i$ and $\varepsilon_{\parallel} = 4.64 - 0.07i$. By using $30nm$ layers of calcium carbonate, and $20nm$ layers of the gain and loss, the total system constitutes a complex birefringent meta-material with $\varepsilon_{xx} = 3.454 - 0.028i$, $\varepsilon_{yy} = 3.453 + 0.028i$, and $\varepsilon_{xy} = 0.038$, which corresponds to $\tau = 0.75$. The initial signal polarizations are separated by $16.2°$ in polarization space, which corresponds to $32.4°$ on the Poincaré sphere (cyan circles). The polarization states at the end of the 'on' unit cells are nearly orthogonal (cyan triangles), and the final output polarization states (cyan squares) are still orthogonal. The surrounding medium is air, $\varepsilon = 1$. (c) Similar to (b), except with an isotropic gain of $\varepsilon = 4 - 0.06i$ and an isotropic loss of $\varepsilon = 9.74 + 0.38i$, and $42.5\mu m$ of the unit cells are 'on,' while $32.5\mu m$ of the unit cells are 'off.' The calcium carbonate layers have remained unchanged. This yields $\tau = 0.44$, and yields orthogonal output states for input signals separated by $40.5°$ in polarization space. The surrounding medium is air, $\varepsilon = 1$.

where the system reverts to being an ordinary birefringent material. If we design the system such that each unit cell consisting of one calcium carbonate layer and one gain/loss comb layer can have its ε_i tuned independently of other unit cells, each individual unit cell can be switched 'on' or 'off,' by setting $\varepsilon_i \neq 0$ or $\varepsilon_i = 0$, respectively, shown in Fig. S2(a). In this way, the total dielectric tensor of the single unit cell in the effective medium limit either has the form

$$\bar{\varepsilon}_{on} = \begin{pmatrix} \varepsilon_r - i\varepsilon_i & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_r + i\varepsilon_i \end{pmatrix}, \tag{S33}$$

or

$$\bar{\varepsilon}_{off} = \begin{pmatrix} \varepsilon_r & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_r \end{pmatrix}. \tag{S34}$$

Here, we have assumed that $\mathrm{Im}[\varepsilon_{gain/loss}] \ll \mathrm{Re}[\varepsilon_{gain/loss}]$, so that the changes in the gain and loss of the semiconductor regions do not affect ε_r. The ability to change ε_i electrically means that it is possible to change the value of $\tau = \varepsilon_i/\varepsilon_{xy}$ in the on unit cells. Similarly, the ability to switch on or off different numbers of unit cells in the meta-material means that the total length of the complex birefringent portion of the meta-material can be changed, i.e. l is electrically tunable.

Now that we can control these two degrees of freedom electrically, this system is a tunable complex birefringent meta-material. Those unit cells which are switched on serve to change the relative polarization between two incident signals to be orthogonal, while those unit cells which are switched off do not change the final output states. This is because the eigen-polarizations of ε_{off} are $|1, \pm 1\rangle$, which are exactly the output states from the complex birefringent portion of the system. To demonstrate this tunability, we consider a meta-material which can transform different pairs of initial polarization states to orthogonally polarized output states, as shown in Fig. S2(b) and (c). As can be seen, the same underlying system with different lengths of on and off sections, as well as different amounts of gain and loss, can produce orthogonal output states from $|\pm 16.2°\rangle$ as well as $|\pm 40.5°\rangle$. Moreover, the presence of the off sections of the meta-material is not seen to effect the final output states. As such, this scheme realizes a tunable complex birefringent meta-material.

V. COMPLEX BIREFRINGENCE WITH ONLY POLARIZATION-DEPENDENT LOSS

As discussed in the main text, by choosing $\varepsilon_{xx} = \varepsilon_{yy}^*$, complex birefringent meta-materials are able to maintain real propagation wavevectors for $|\tau| < 1$. However, in many experimental systems it may not be possible to achieve such a perfect balance of gain and loss. Despite this, many systems for which $\mathrm{Im}[\varepsilon_{xx}] \neq -\mathrm{Im}[\varepsilon_{yy}]$ still provide nearly arbitrary control over the polarization. The most important of these cases is a system with only polarization-dependent loss,

$$\varepsilon_{xx} = \varepsilon_r, \tag{S35}$$

$$\varepsilon_{yy} = \varepsilon_r + 2i\varepsilon_i. \tag{S36}$$

In such systems both allowed wavevectors become complex,

$$\frac{k_\pm^2}{\omega^2\mu} = \varepsilon_r + i\varepsilon_i \pm \varepsilon_{xy}\sqrt{1-\tau^2}, \tag{S37}$$

leading to attenuation of the signal, causing the polarization states to eventually decay to $|E_+^R\rangle$. Fortunately, for modest values of the polarization dependent loss, significant control over the polarization can still be obtained. For example, in Fig. S3, $\varepsilon_r = 1.1$, $\varepsilon_{xy} = 0.1$, and $\varepsilon_i = 0.05$, and as can be seen, there is only a small overall shift in the polarizations towards $|E_+^R\rangle$ for a complete rotation about the Poincaré sphere. Thus, such complex birefringent meta-materials with additional loss can still enable nearly arbitrary control over pairs of polarization states, as such materials can transform two initial states $|\pm\theta\rangle$ to $\approx |1, \pm 1\rangle$ in an analogous manner to the complex birefringent meta-materials discussed in the main text. Here, the final states will not be exactly orthogonal, as the polarization of both incident signals has been shifted slightly towards $|E_+^R\rangle$ relative to what one would find in a complex birefringent meta-material with $\varepsilon_{xx} = \varepsilon_{yy}^*$. We note that as ε_{xy} decreases and the required propagation distance to produce orthogonal polarization states increases, the absorption in Eq. (S37) becomes a more important effect, and can eventually result in the inability of the meta-material to provide arbitrary polarization control.

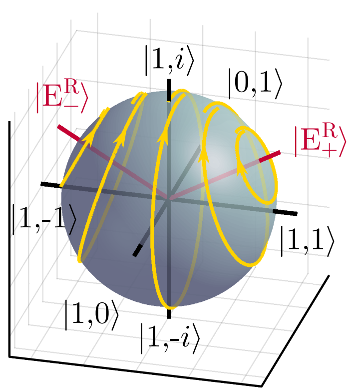

FIG. S3. Polarization evolution in complex birefringent materials with only loss. Flow lines (yellow) depict the evolution of the polarization when traversing through a complex birefringent material with $\tau = 0.5$ which only has polarization-dependent loss. The eigenvectors of the dielectric tensor are shown in red. Here, $\varepsilon_{xy} = 0.1$ and $\varepsilon_i = 0.05$ are used.

VI. OFF-NORMAL PROPAGATION THROUGH COMPLEX BIREFRINGENT META-MATERIALS

In traditional, lossless birefringent media, finding the propagation constants of the system is typically done by first expressing $\bar{\varepsilon}$ along the material's principal axes, then rotating the tensor such that one of the axes aligns with the propagation direction and solving the eigenvalue equation [5],

$$\bar{\kappa}\mathbf{D} = \lambda\mathbf{D}, \tag{S38}$$

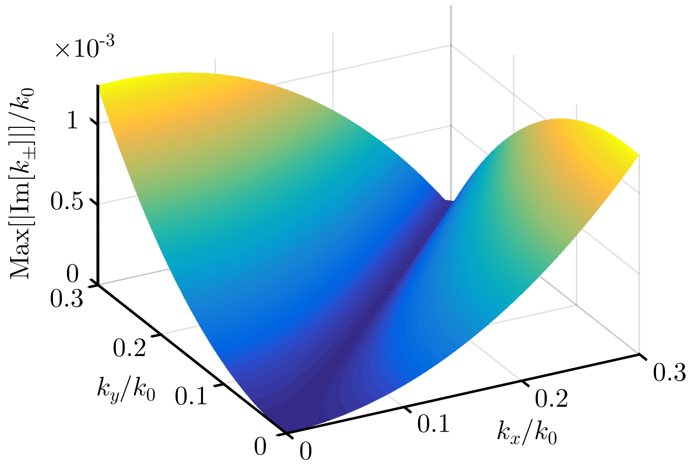

FIG. S4. Acquisition of imaginary component of propagation coefficients for off-axis propagation. Plot of the maximum value of $|\mathrm{Im}[k_{\pm}]|$ as k_x and k_y are increased relative to $k_0 = \omega\sqrt{\varepsilon_r \mu}$. Here, $\varepsilon_r = 2$, $\varepsilon_i = 0.1$, and $\varepsilon_{xy} = 0.2$.

where $\mathbf{D}$ is the electric displacement field, and $\bar{\kappa} = \bar{\varepsilon}^{-1}$. As $\mathbf{D}$ must always be perpendicular to $\mathbf{k}$, performing this rotation means that Eq. (S38) is only a 2×2 system of equations which can be solved for $\lambda_{\pm}$, which are related to the propagation constants as $k_{\pm} = \omega\sqrt{\mu/\lambda_{\pm}}$. Unfortunately, this method is not immediately applicable to complex birefringent meta-materials, as the principal axes of such systems are not orthogonal. Thus, the diagonal form of $\bar{\kappa}$ is not related to any choice of orthogonal coordinate system via a unitary similarity transformation.

It is still possible to solve for the off-axis propagation coefficients by directly solving the wave equation as [6],

$$\left[\mathbf{k} \times \mathbf{k} \times -\omega^2 \mu \bar{\varepsilon}\right] \mathbf{E} = 0. \tag{S39}$$

Here, $\mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$, and k_x and k_y are specified as inputs describing the off-normal propagation direction. Then, Eq. (S39) is solved for the allowed values of k_z. In general, solving Eq. (S39) will lead to a quartic equation for k_z, with the four eigenvalues corresponding to forwards and backwards traveling waves with wavevectors $k_{\pm}$.

In a complex birefringent material, one can intuitively understand the real propagation coefficients, $k_{\pm} \in \mathbb{R}$, when $|\tau| < 1$ as due to the wave spending equal time in both the gain and loss regions of the material. As such, we expect this balance to be disrupted for off-normal propagation through such materials, leading to complex propagation coefficients, $k_{\pm} \in \mathbb{C}$ even when $|\tau| < 1$. However, as can be seen in Fig. S4, the rate at which the propagation coefficients become complex is slow, even for the relatively large value of $\varepsilon_i = 0.1$ used here, along with $\varepsilon_{xy} = 0.2$ and $\varepsilon_r = 2$. Furthermore, in propagation directions which preserve the balance of gain and loss in the system, $|k_x| = |k_y|$, the propagation coefficients remain real.

VII. POLARIZATION FLOW IN THE ABSENCE OF INDEX-MATCHING

The polarization flow diagrams shown in the main text in Figs. 1 and 2 are calculated assuming that the complex birefringent meta-material is index-matched to the surrounding medium, such that $\varepsilon_r = \varepsilon_{surround}$. When this criteria is not met, the general behavior of the complex birefringent meta-material is still similar, as can be seen in Fig. S5, but the polarization flow no longer revolves around the Poincaré sphere in a monotonic fashion as the effective propagation length, $k_{xy}l$ is increased. However, this does not prevent the complex birefringent meta-material from enabling arbitrary polarization conversion.

---

[1] L. Ge, Y. D. Chong, and A. D. Stone, Phys. Rev. A **85**, 023802 (2012).
[2] C. Ye, Opt. Eng **34**, 3031 (1995).
[3] Z. Zhuang, S.-W. Suh, and J. S. Patel, Opt. Lett. **24**, 694 (1999).
[4] A. Safrani and I. Abdulhalim, Opt. Lett. **34**, 1801 (2009).
[5] H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice Hall, Englewood Cliffs, NJ, 1983).
[6] P. Yeh, *Optical Waves in Layered Media* (Wiley, 2005).

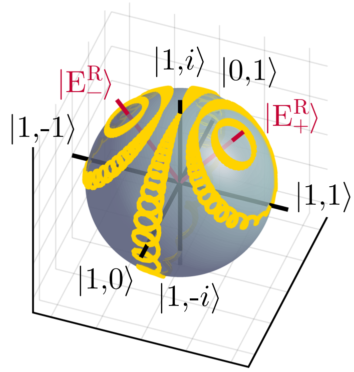

FIG. S5. Polarization evolution in complex birefringent materials when the material is not index-matched with the surrounding medium. Flow lines (yellow) depict the evolution of the polarization when traversing through a complex birefringent material with $\tau = 0.81$, $\varepsilon_r = 3$, $\varepsilon_{xy} = 0.25$, and surrounded by air, $\varepsilon_{air} = 1$. The eigenvectors of the dielectric tensor are shown in red.