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Multivariable pseudospectrum in C^*-algebras

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ABSTRACT

We look at various forms of spectrum and associated pseudospectrum that can be defined for noncommuting d-tuples of Hermitian elements of a C^*-algebra. In particular, we focus on the forms of multivariable pseudospectra that are finding applications in physics. The emphasis is on theoretical calculations of examples, in particular for noncommuting pairs and triple of operators on infinite dimensional Hilbert space. In particular, we look at the universal pair of projections in a C^*-algebra, the usual position and momentum operators, and triples of tridiagonal operators. We prove a relation between the quadratic pseudospectrum and Clifford pseudospectra, as well as results about how symmetries in a tuple of operators can lead to a symmetry in the various pseudospectra.

1. Introduction

Given $\mathbb{A} = (A_1, \ldots, A_d)$, a noncommuting d-tuple of bounded linear Hermitian operators on Hilbert space, so $A_j = A_j^* \in \mathcal{B}(\mathcal{H})$, there are many competing notions of a joint spectrum. The one that has been involved in recent developments in photonics [3,7], metamaterials [10] and condensed matter physics [27], is the Clifford spectrum $\Lambda(\mathbb{A})$ and the more general Clifford pseudospectrum. What sets this apart from other definitions of joint pseudospectrum is that it naturally leads to fast numerical algorithms that apply to finite models of quantum materials. Here we look at computing the Clifford spectrum and pseudospectrum of d-tuples of collections of operators, some related to weighted-shifts, on separable Hilbert space.

The Clifford spectrum is a closed, bounded subset of $\mathbb{R}^d$. To define it, we define first the *spectral localizer*

$$L_{\boldsymbol{\lambda}}(\mathbb{A}) = \sum_{j=1}^{d} (A_j - \lambda_j I) \otimes \Gamma_j .$$

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Here $(\Gamma_1, \ldots, \Gamma_d)$ are matrices that satisfy the relations

$$\Gamma_j^* = \Gamma_j, \quad (j = 1, \ldots, d)$$
$$\Gamma_j^2 = 1, \quad (j = 1, \ldots, d)$$
$$\Gamma_j \Gamma_k = -\Gamma_k \Gamma_j, \quad (j \neq k)$$

and we refer to these as a *representation of the Clifford relations*. It is important that the spectral localizer is Hermitian, at least in this context where the A_j are all Hermitian. In physics, this assumption is sometimes dropped [7,13], but we stick with the Hermitian setting here.

It does not matter what representation we use [4, Lemma 1.2], so generally we take an irreducible representation, which means these matrices are of size $2^{\lfloor d/2 \rfloor}$. For $d = 2$ a standard choice is

$$\Gamma_1 = \sigma_x, \, \Gamma_2 = \sigma_y \tag{1.1}$$

and for $d = 3$ the standard choice is

$$\Gamma_1 = \sigma_x, \, \Gamma_2 = \sigma_y, \, \Gamma_3 = \sigma_z. \tag{1.2}$$

Here we are using the Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The *Clifford spectrum* is then

$$\Lambda^{\mathrm{C}}(\mathbb{A}) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^d \middle| L_{\boldsymbol{\lambda}}(\mathbb{A}) \text{ is not invertible} \right\}. \tag{1.3}$$

We will see that an interesting subset of this is the *essential Clifford spectrum*, defined as

$$\Lambda^{\mathrm{C}}_{\mathrm{e}}(\mathbb{A}) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^d \middle| L_{\boldsymbol{\lambda}}(\mathbb{A}) \text{ is not Fredholm} \right\}.$$

From a mathematical point of view, there is much that is unknown about the Clifford spectrum. We are not even sure if it can be the empty set [4, §8], although a lot of evidence says this does not happen. Most of the examples examined so far have required computer calculations, so most examples have been finite-dimensional. The need for a computer to assist in calculations is not surprising, as the Clifford pseudospectrum was designed to work well in computer models of quantum systems.

Here we move on to infinite-dimensional examples. In most cases, we work inside the Toeplitz algebra, so that we work with easy C^*-algebras inside the Calkin algebra to calculate most of the Clifford spectrum, and then reduce the search for the rest of the Clifford spectrum to a calculation involving some difference equations. We are then able to find infinite-dimensional examples where the Clifford spectrum looks very different from the Clifford spectra that arise from similar examples in finitely many dimensions.

If $\mathbb{A} = (A_1, \ldots, A_d)$, where each A_j is an Hermitian element of a unital C^*-algebra $\mathcal{A}$, the definition of Clifford spectrum still makes sense. Taking the Γ_j to be matrices in $M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C})$, we treat $L_{\boldsymbol{\lambda}}(\mathbb{A})$ as an element of $M_{2^{\lfloor d/2 \rfloor}}(\mathcal{A}) \cong \mathcal{A} \otimes M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C})$. We now are treating I as the identity element of $\mathcal{A}$. As $\Lambda(\mathbb{A})$ is defined in Eq. (1.3), in term of invertiblity, we find we have spectral permanence. That is, if $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{B}$, then we get the same result if we compute $\Lambda(\mathbb{A})$ working in $M_{2^{\lfloor d/2 \rfloor}}(\mathcal{B})$ as we do if working in $M_{2^{\lfloor d/2 \rfloor}}(\mathcal{A})$.

2. Symmetries and $*$-homomorphisms

We collect here some basic lemmas on the Clifford spectrum in C^*-algebras. We already discussed spectral permanence, so we already know the behavior of the Clifford spectrum with respect to an embedding of C^*-algebras.

Theorem 2.1. *Suppose that $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism between unital C^*-algebras. If $A_1, \ldots, A_d$ are Hermitian elements of $\mathcal{A}$ then*

$$\Lambda^{\mathrm{C}}(\varphi(A_1), \ldots, \varphi(A_d)) \subseteq \Lambda^{\mathrm{C}}(A_1, \ldots, A_d). \tag{2.1}$$

If φ is one-to-one, then the inclusion in Equation (2.1) becomes an equality.

Proof. Since

$$(\varphi \otimes I)\left(L_{\boldsymbol{\lambda}}\left(A_1, \ldots, A_d\right)\right) = L_{\boldsymbol{\lambda}}\left(\varphi(A_1), \ldots, \varphi(A_d)\right),$$

we know that if $L_{\boldsymbol{\lambda}}\left(A_1, \ldots, A_d\right)$ is invertible then $L_{\boldsymbol{\lambda}}\left(\varphi(A_1), \ldots, \varphi(A_d)\right)$ is also invertible. □

Corollary 2.2. *If $A_1, \ldots, A_d$ in $\mathcal{B}(\mathcal{H})$ are Hermitian and U is a unitary operator on $\mathcal{H}$ then*

$$\Lambda^{\mathrm{C}}(UA_1U^*, \ldots, UA_dU^*) = \Lambda^{\mathrm{C}}(A_1, \ldots, A_d)$$

and

$$\Lambda^{\mathrm{C}}_{\mathrm{e}}(UA_1U^*, \ldots, UA_dU^*) = \Lambda^{\mathrm{C}}_{\mathrm{e}}(A_1, \ldots, A_d).$$

For many purposes, such as proving that a symmetry in $\mathbb{A}$ leads to a symmetry in $\Lambda^{\mathrm{C}}(\mathbb{A})$, we need to know that we have complete flexibility in selecting the Γ_j.

Lemma 2.3. *Suppose $\Gamma_1, \ldots, \Gamma_d$ form a representation of the Clifford relations in $\boldsymbol{M}_r(\mathbb{C})$ and $\Gamma'_1, \ldots, \Gamma'_d$ form a representation of the Clifford relations in $\boldsymbol{M}_s(\mathbb{C})$. If $A_1, \ldots, A_d$ are Hermitian elements of unital C^*-algebra $\mathcal{A}$ then*

$$\sum_{j=1}^{d} (A_j - \lambda_j) \otimes \Gamma_j$$

is invertible if, and only if,

$$\sum_{j=1}^{d} (A_j - \lambda_j) \otimes \Gamma'_j$$

is invertible.

The proof of this lemma is essentially the same as the proof of [4, Lemma 1.2] and is omitted. We need also the following lemmas from [4], also generalized to the C^*-algebra setting. Again the proofs are almost identical to the matrix case and are omitted. We will denote by $O(d)$ the real-valued orthogonal matrices of size d.

Lemma 2.4. *Suppose* $(A_1, \ldots, A_d)$ *is a d-tuple of Hermitian elements of unital* C^**-algebra* $\mathcal{A}$ *and that* $U \in O(d)$. *Suppose* $\boldsymbol{\lambda} \in \mathbb{R}^d$. *The d elements*

$$\hat{A}_j = \sum_s u_{js} A_s$$

are also Hermitian and

$$\boldsymbol{\lambda} \in \Lambda^{\mathrm{C}}(A_1, \ldots, A_d) \iff U\boldsymbol{\lambda} \in \Lambda^{\mathrm{C}}(\hat{A}_1, \ldots, \hat{A}_d).$$

Theorem 2.5. *Suppose* $(A_1, \ldots, A_d)$ *are Hermitian elements in the unital* C^**-algebra* $\mathcal{A}$ *and that* $U \in O(d)$. *Let*

$$\hat{A}_j = \sum_s u_{js} A_s.$$

If there exists a unitary Q *in* $\mathcal{A}$ *such that* $Q\hat{A}_j Q^* = A_j$ *for all* j *then*

$$\boldsymbol{\lambda} \in \Lambda^{\mathrm{C}}(A_1, \ldots, A_d) \iff U\boldsymbol{\lambda} \in \Lambda^{\mathrm{C}}(A_1, \ldots, A_d).$$

3. The commuting and essentially commuting cases

We know that a single operator's spectrum can look very different in the infinite-dimensional case when compared with the spectrum of a finite-dimensional counterpart. In the finite-dimensional case, any operator (or matrix) T will have finite spectrum. In contrast, in infinite dimensions we can get any nonempty closed and bounded subset of the complex plane. This phenomenon will give us our first examples where the Clifford spectrum looks different from how things looked in finite dimensions [12,31].

In the case $d = 2$ the Clifford spectrum is a minor variation on the ordinary spectrum. With the standard Γ matrices, as in Eq. (1.1), the spectral localizer is

$$L_{(x,y)}(A_1, A_2) = \begin{bmatrix} 0 & A_1 - iA_2 - (x - iy)I \\ A_1 + iA_2 - (x + iy)I & 0 \end{bmatrix}.$$

This tells us immediately that $(x, y) \in \Lambda^{\mathrm{C}}(A_1, A_2)$ exactly when $x + iy$ is in the ordinary spectrum of $A_1 + iA_2$. For example, we can have $A_1 + iA_2$ be the bilateral shift and so have an example where

$$\Lambda^{\mathrm{C}}(A_1, A_2) = \mathbb{T}^1. \tag{3.1}$$

In finite dimensions, we have conjectured [4, §8] that one-dimensional manifolds cannot arise as the Clifford spectrum of three matrices. In infinite dimensions, we can get the circle as the Clifford spectrum of three operators with commutative and noncommutative examples.

Lemma 3.1. *If* $A_1, \ldots, A_d$ *are pair-wise commuting Hermitian elements of a unital* C^**-algebra* $\mathcal{A}$ *then the Clifford spectrum of* $\mathbb{A} = (A_1, \ldots, A_d)$ *equals the standard joint spectrum.*

Proof. Because of spectral permanence and Corollary 2.2, we can assume that $\mathcal{A} = C(X)$ for some compact Hausdorff space and that $A_j = f_j$ for some continuous $f_j : X \to \mathbb{R}$. In this case, we can work pointwise and we find

$$L_{\boldsymbol{\lambda}}(\mathbb{A}) = g : X \to M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C})$$

where

$$g(x) = \sum_{j=1}^{d} (f_j(x) - \lambda_j)\Gamma_j.$$

An easy calculation shows that, for scalars α_j,

$$\sigma\left(\sum \alpha_j \Gamma_j\right) = \left\{\pm\sqrt{\sum \alpha_j^2}\right\}.$$

Thus g is singular only when there is a point x_0 in X such that $f_j(x_0) = \lambda_j$ for all j. Thus the Clifford spectrum is just the joint spectrum. □

Example 3.2. If X is a compact nonempty subset of $\mathbb{R}^d$ then there is a commutative example of d Hermitian operators on separable Hilbert space whose Clifford spectrum equals X. Notice that X is metrizable so $C(X)$ is separable and so can be represented on a separable Hilbert space.

Theorem 3.3. *Suppose $A_1, \ldots, A_d$ are Hermitian elements of a unital C^*-algebra $\mathcal{A}$ and that $1 \leq r < d$. If A_j commutes with A_k whenever $j \leq r$ and $k > r$ then*

$$\Lambda^{\mathrm{C}}(A_1, \cdots, A_d) \subseteq \Lambda^{\mathrm{C}}(A_1, \cdots, A_r) \times \Lambda^{\mathrm{C}}(A_{r+1}, \cdots, A_d).$$

Proof. We always have

$$(L_{\boldsymbol{\lambda}}(A_1, \cdots, A_d))^2 = \sum (A_j - \lambda_j)^2 \otimes I + \sum_{j<k} [A_j, A_k] \otimes \Gamma_j \Gamma_k \tag{3.2}$$

but with the given assumptions many commutators vanish. Here we obtain

$$(L_{\boldsymbol{\lambda}}(A_1, \cdots, A_d))^2 = \sum_{j=1}^{d} (A_j - \lambda_j)^2 \otimes I + \sum_{j<k\leq r} [A_j, A_k] \otimes \Gamma_j \Gamma_k + \sum_{r<j<k} [A_j, A_k] \otimes \Gamma_j \Gamma_k$$

which implies

$$(L_{\boldsymbol{\lambda}}(A_1, \cdots, A_d))^2 = \left(L_{(\lambda_1,\ldots,\lambda_r)}(A_1, \cdots, A_r)\right)^2 + \left(L_{(\lambda_{r+1},\ldots,\lambda_d)}(A_{r+1}, \cdots, A_d)\right)^2. \tag{3.3}$$

If $(\lambda_1, \ldots, \lambda_r) \notin \Lambda^{\mathrm{C}}(A_1, \cdots, A_r)$ then there is a positive a such that

$$a \leq \left(L_{(\lambda_1,\ldots,\lambda_r)}(A_1, \cdots, A_r)\right)^2.$$

We always have

$$0 \leq \left(L_{(\lambda_{r+1},\ldots,\lambda_d)}(A_{r+1}, \cdots, A_d)\right)^2$$

and so

$$a \leq (L_{\boldsymbol{\lambda}}(A_1, \cdots, A_d))^2.$$

Thus

$$(\lambda_1, \ldots, \lambda_r) \notin \Lambda^{\mathrm{C}}(A_1, \cdots, A_r) \implies (\lambda_1, \ldots, \lambda_d) \notin \Lambda^{\mathrm{C}}(A_1, \cdots, A_d).$$

By symmetry,

$$(\lambda_{r+1},\ldots,\lambda_d)\notin\Lambda^{\mathrm{C}}(A_{r+1},\cdots,A_d)\implies(\lambda_1,\ldots,\lambda_d)\notin\Lambda^{\mathrm{C}}(A_1,\cdots,A_d).\quad\square$$

Notice that the reverse inclusion in Theorem 3.3 is already false in the case where all the A_j commute. We do get equality in a simple special case.

Theorem 3.4. *Suppose $A_1,\ldots,A_{d-1}$ are Hermitian elements of a unital C^*-algebra $\mathcal{A}$. For any real scalar α we have*

$$\Lambda^{\mathrm{C}}(A_1,\cdots,A_{d-1},\alpha I)=\Lambda^{\mathrm{C}}(A_1,\cdots,A_{d-1})\times\{\alpha\}.$$

Proof. Equation (3.3) here becomes

$$(L_{\boldsymbol{\lambda}}(A_1,\cdots,A_{d-1},\alpha I))^2=\left(L_{(\lambda_1,\ldots,\lambda_{d-1})}(A_1,\cdots,A_{d-1})\right)^2+(\alpha-\lambda_d)^2\,I$$

and the result follows. $\square$

The following provides more evidence to support two conjectures from [4], that when the Clifford spectrum of a d-tuple of n-by-n matrices is nonempty, and that when it is finite, it must have cardinality at most n.

Theorem 3.5. *If A_1, A_2 and A_3 are n-by-n Hermitian matrices and A_1 commutes with both A_2 and A_3 then the Clifford spectrum of (A_1,A_2,A_3) is a set containing at least one point and at most n points.*

Proof. First we look at the special case where the first matrix is scalar. Here Theorem 3.4 tells us that $\Lambda^{\mathrm{C}}(\alpha I,A_2,A_3)$ equals $\{\alpha\}\times\sigma(A_2+iA_3)$ which is nonempty and can have no more than n points.

In the general case, let $\alpha_1,\ldots,\alpha_r$ denote the distinct eigenvalues of A_1. We can conjugate all the matrices by a unitary to ensure that

$$A_1=\begin{bmatrix}\alpha_1 I & & \\ & \ddots & \\ & & \alpha_r I\end{bmatrix}.$$

The fact that the other matrices commute with A_1 forces them to be block diagonal,

$$A_2=\begin{bmatrix}B_1 & & \\ & \ddots & \\ & & B_r\end{bmatrix},\quad A_3=\begin{bmatrix}C_1 & & \\ & \ddots & \\ & & C_r\end{bmatrix}.$$

Therefore

$$\Lambda^{\mathrm{C}}(A_1,A_2,A_3)=\bigcup_{j=1}^{r}\Lambda^{\mathrm{C}}(\alpha_j I,B_j,C_j)$$

and the result follows. $\square$

4. Singular values in infinite dimensions

The spectrum of a matrix is generally less informative when that matrix is not normal. In essence, this is why we cannot build as wide-ranging functional calculus in the nonnormal case as in the normal case. Some additional information can be found in the pseudospectrum of a nonnormal matrix. The pseudospectrum [32] of a square matrix X is based on look formally at the function

$$\alpha \mapsto \left\| (X - \alpha I)^{-1} \right\|^{-1} \tag{4.1}$$

which takes on a gradation of values for $\alpha \in \mathbb{C}$, including 0 by default when $X - \alpha$ is not invertible. This function can be seen as the pseudospectrum, but traditionally one looked at the inverse image of $[0, \epsilon)$ and called that the ϵ-pseudospectrum.

In applied math, the pseudospectrum is only defined for a single, typically non-normal, square matrix. One can easily translate this into a theory applying to two Hermitian matrices by considering $X = A_1 + iA_2$. The norm in Eq. (4.1) can be interpreted in many ways. Here we are only interested in the operator norm. We will find it more convenient to compute $\|X^{-1}\|^{-1}$ as the smallest singular value of a matrix. Various papers have looked at the pseudospectrum of a single nonnormal operator on separable Hilbert space, including [2,16,18] in operator theory and [20] in physics.

We need a replacement for the smallest singular value of a non-square matrix that works for a bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$. For now, we are content to deal with the bounded linear operators on separable Hilbert space. In the finite-dimensional case, one characterization of the smallest singular value is the minimum value of $\|T\boldsymbol{v}\|$ as $\boldsymbol{v}$ ranges over all unit vectors. We take this as a definition in the infinite-dimensional case, except we use infimum,

$$s_{\min}(T) = \inf_{\|\boldsymbol{v}\|=1} \|T\boldsymbol{v}\|.$$

In the special case of a compact operator, there is a spectral decomposition [15] and so all singular values are defined. We need only the smallest, and of course the largest since we are working with the spectral norm.

The following lemma allows us to extend the definition of $s_{\min}$ to normal elements of a C^*-algebra. We can thus extend the definition of the Clifford pseudospectrum for the C^*-algebra setting. Before we make that definition we present some basic results about $s_{\min}$ of operators. We claim no originality, but could not find the exact results we needed in the literature.

Lemma 4.1. *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible, then*

$$s_{\min}(T) = \|T^{-1}\|^{-1}.$$

Proof. By homogeneity we can compute $s_{\min}(T)$ as

$$s_{\min}(T) = \inf_{\boldsymbol{v} \neq 0} \frac{\|T\boldsymbol{v}\|}{\|\boldsymbol{v}\|}.$$

If $T\boldsymbol{v} = \boldsymbol{w}$ then

$$\left(\frac{\|T\boldsymbol{v}\|}{\|\boldsymbol{v}\|} \right)^{-1} = \frac{\|T^{-1}\boldsymbol{w}\|}{\|\boldsymbol{w}\|}$$

and the result now follows. □

Lemma 4.2. *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator. If T is normal, then*

$$s_{\min}(T) = \min\left\{ |\lambda| \,\middle|\, \lambda \in \sigma(T) \right\}. \tag{4.2}$$

Proof. We know

$$\inf_{\|\mathbf{v}\|=1} \|T\mathbf{v}\| = \inf_{\|\mathbf{v}\|=1} \|(T^*T)^{-\frac{1}{2}}\mathbf{v}\| = \operatorname{dist}(0, \sigma((T^*T)^{-\frac{1}{2}})).$$

On the other hand, applying to spectral mapping theorem to the normal operator T and continuous function $|\lambda|$ we find

$$\sigma((T^*T)^{-\frac{1}{2}}) = \{|\lambda| : \lambda \in \sigma(T)\}$$

and the result follows. □

Lemma 4.3. *Suppose $S, T : \mathcal{H}_1 \to \mathcal{H}_2$ are both bounded linear operators, then*

$$|s_{\min}(S) - s_{\min}(T)| \leq \|S - T\|.$$

Proof. Let $\epsilon > 0$ be given. Then there is a unit vector $\boldsymbol{v}$ so that

$$\|T\boldsymbol{v}\| \leq s_{\min}(T) + \epsilon.$$

Then

$$s_{\min}(S) \leq \|S\boldsymbol{v}\| \leq \|T\boldsymbol{v}\| + \|(S-T)\boldsymbol{v}\| \leq s_{\min}(T) + \epsilon + \|S - T\|$$

proving

$$s_{\min}(S) - s_{\min}(T) \leq \|S - T\| + \epsilon.$$

As this is true for all $\epsilon > 0$, we have shown

$$s_{\min}(S) - s_{\min}(T) \leq \|S - T\|.$$

We are done, by symmetry. □

Remark 4.4. Unlike the case in finite dimensions, $s_{\min}(T) \neq 0$ does not always imply T is invertible. However, when $T^* = T$ it is true that $s_{\min}(T) \neq 0$ if and only if T^{-1} exists.

Lemma 4.5. *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_2 \to \mathcal{H}_3$ are bounded linear operators then*

$$s_{\min}(ST) \geq s_{\min}(S)s_{\min}(T)$$

Proof. Recall that

$$s_{\min}(T) = \inf_{\boldsymbol{v} \neq 0} \frac{\|T\boldsymbol{v}\|}{\|\boldsymbol{v}\|}.$$

If T has no kernel then, for every nonzero $\boldsymbol{v}$, we have

$$\frac{\|ST\boldsymbol{v}\|}{\|\boldsymbol{v}\|}=\frac{\|ST\boldsymbol{v}\|}{\|T\boldsymbol{v}\|}\frac{\|T\boldsymbol{v}\|}{\|\boldsymbol{v}\|}\geq s_{\min}(S)s_{\min}(T)$$

so

$$s_{\min}(ST)\geq s_{\min}(S)s_{\min}(T).$$

If T has a kernel then ST has a kernel so both $s_{\min}(ST)$ and $s_{\min}(T)$ are zero and we are done for trivial reasons. □

One might think that T and T^* and

$$\begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}$$

all have the same $s_{\min}$. After all, this is true in the finite-dimensional case. However, if we let T be the forward shift, then $s_{\min}(T)=1$ since $\|T\boldsymbol{v}\|=\|\boldsymbol{v}\|$ for all $\boldsymbol{v}$ (that is, T is an isometry). In contrast, there is a unit vector $\boldsymbol{v}$ so that the backwards shift sends $\boldsymbol{v}$ so zero, so that $\|T^*\boldsymbol{v}\|=0$. Thus also

$$\begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so we can conclude

$$s_{\min}\begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}=s_{\min}(T^*)=0.$$

Finally, notice that $\sqrt{T^*T}=I$ while $\sqrt{TT^*}$ has a kernel, so

$$s_{\min}\left(|T^*|\right)\neq s_{\min}\left(|T|\right)$$

in infinite dimensions.

For two Hilbert spaces $\mathcal{H}_i,\ i=1,2$, let $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ denote the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Recall that for $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$, $|T|\in\mathcal{L}(\mathcal{H}_1)$ denotes the modulus of T, that is the unique positive semi-definite operator so that $|T|^2=T^*T$ and $\ker(T)=\ker(|T|)$.

Lemma 4.6. *If $T:\mathcal{H}_1\to\mathcal{H}_2$ is a bounded linear operator then*

$$s_{\min}(T)=s_{\min}(|T|)=\sqrt{s_{\min}(T^*T)}.$$

Proof. Since $\|T\boldsymbol{v}\|=\|\,|T|\boldsymbol{v}\,\|$ for all $\boldsymbol{v}\in\mathcal{H}_1$, the consequence that $s_{\min}(T)=s_{\min}(|T|)$ is straightforward. The equality of $s_{\min}(|T|)$ with $\sqrt{s_{\min}(T^*T)}$ follows from Lemma 4.2 and the spectral mapping theorem. □

Definition 4.7. Suppose $A_1,\dots,A_d$ are Hermitian elements of a unital C^*-algebra $\mathcal{A}$. The *Clifford pseudospectrum* for $\mathbb{A}=(A_1,\dots,A_d)$ is the function $\boldsymbol{\lambda}\mapsto\mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(\mathbb{A})$ where

$$\mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(\mathbb{A})=s_{\min}\left(L_{\boldsymbol{\lambda}}(\mathbb{A})\right).$$

Here we use Equation (4.2) to define $s_{\min}$ of a normal element, here the spectral localizer, in a C^*-algebra.

Once again, we can use the argument given in [4, Lemma 1.2] to see that $\mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(\mathbb{A})$ does not depend on the Clifford representation used to form the spectral localizer. Notice that the Clifford spectrum is the null-set of the Clifford pseudospectrum,

$$\Lambda^{\mathrm{C}}(\mathbb{A}) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^d \middle| \, \mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(\mathbb{A}) = 0 \right\}.$$

We can now extend some of the result on Clifford spectrum and pseudospectrum that have been previously worked out in the case of finite matrices [6,23]. Theorem 4.8 is a generalization of Lemma 2.3 of [28]. That result handles one general operator, or equivalently two Hermitian operators.

Theorem 4.8. *Suppose $A_1, \ldots, A_d$ are Hermitian elements of a unital C^*-algebra $\mathcal{A}$. The Clifford pseudospectrum of $\mathbb{A}$ is Lipschitz, specifically with*

$$\left|\mu^{C}_{\boldsymbol{\lambda}}(A_1, \ldots, A_d) - \mu^{C}_{\boldsymbol{\nu}}(A_1, \ldots, A_d)\right| \leq \|\boldsymbol{\lambda} - \boldsymbol{\nu}\| \tag{4.3}$$

where the norm is taken to be the Euclidean norm on $\mathbb{R}^n$. Moreover

$$\left|\mu^{C}_{\boldsymbol{\lambda}}(\mathbb{A}) - \|\boldsymbol{\lambda}\|\right| \leq \|L_{\mathbf{0}}(\mathbb{A})\|. \tag{4.4}$$

Proof. These statements all follow from the fact that $s_{\min}(A)$ is Lipschitz in the operator norm (Lemma 4.3). This result is a generalization of results in Section 7 of [23], and is proven by related methods. The essential calculations are

$$\sigma\left(L_{\mathbf{0}}\left(\lambda_1 I, \cdots, \lambda_d I\right)\right) = \left\{\|\boldsymbol{\lambda}\|, -\|\boldsymbol{\lambda}\|\right\}$$

and its corollary

$$\left\|L_{\mathbf{0}}\left(\lambda_1 I, \cdots, \lambda_d I\right)\right\| = \|\boldsymbol{\lambda}\|.$$

For (4.3) note that

$$L_{\boldsymbol{\lambda}}\left(A_1, \cdots, A_d\right) - L_{\boldsymbol{\nu}}\left(A_1, \cdots, A_d\right) = L_{\mathbf{0}}\left((\nu_1 - \lambda_1)I, \cdots, (\nu_d - \lambda_d)I\right)$$

which implies that

$$\left\|L_{\boldsymbol{\lambda}}\left(A_1, \cdots, A_d\right) - L_{\boldsymbol{\nu}}\left(A_1, \cdots, A_d\right)\right\| = \|\boldsymbol{\lambda} - \boldsymbol{\nu}\|.$$

For (4.4), we know that

$$s_{\min}\left(L_{\mathbf{0}}\left(\lambda_1 I, \cdots, \lambda_d I\right)\right) = \|\boldsymbol{\lambda}\|$$

and

$$\left\|L_{\mathbf{0}}\left(\lambda_1 I, \cdots, \lambda_d I\right) + L_{\boldsymbol{\lambda}}\left(A_1, \cdots, A_d\right)\right\| = \left\|L_{\mathbf{0}}\left(A_1, \cdots, A_d\right)\right\|. \quad \square$$

Corollary 4.9. *The Clifford spectrum for a d-tuple of Hermitian elements of a C^*-algebra is always compact.*

Proof. Since Lipschitz implies continuous we get continuity from Equation (4.3). Equation (4.4) tells us that then $\|\boldsymbol{\lambda}\| > \|L_{\mathbf{0}}(\mathbb{A})\|$ we cannot have $\mu^{C}_{\boldsymbol{\lambda}}(\mathbb{A}) = 0$. Therefore the Clifford spectrum of $(A_1, \ldots, A_d)$ is a closed subset of the ball at the origin of radius $\|L_{\mathbf{0}}(A_1, \ldots, A_d)\|$. $\square$

5. Other forms of multivariable pseudospectrum

5.1. The quadratic pseudospectrum and more

There are many ways to define a pseudospectrum for $\mathbb{A} = (A_1, \ldots, A_d)$ that are Hermitian. If these are operators on Hilbert space $\mathcal{H}$, Mumford [26] suggests associating to $\boldsymbol{\lambda}$ the number

$$\min \left\{ \max \|A_j \boldsymbol{v} - \lambda_j \boldsymbol{v}\| \;\middle|\; \|\boldsymbol{v}\| = 1 \text{ and } \mathrm{E}_{\boldsymbol{v}}[A_j] = \lambda_j \text{ for all } j \right\} \tag{5.1}$$

where $\mathrm{E}_{\boldsymbol{v}}[A_j] = \langle A_j \boldsymbol{v}, \boldsymbol{v} \rangle$ is the expectation for the observable A_j when the system is in state $\boldsymbol{v}$. This is expected [23, §1] to be a very difficult, but important, minimization problem related to joint measurement.

Fortunately, there is a slight modification that can be made to Equation (5.1) that turns it into something very computable, as the minimization problem leads to a number equal to the smallest spectral value of a Hermitian matrix. The *quadratic pseudospectrum* [6] is defined as

$$\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(A_1, \ldots, A_d) = \min \left\{ \sqrt{\sum_j \|A_j \boldsymbol{v} - \lambda_j \boldsymbol{v}\|^2} \;\middle|\; \|\boldsymbol{v}\| = 1 \right\}. \tag{5.2}$$

The practical way to compute this is to use the known equality [6]

$$\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(A_1, \ldots, A_d) = \sqrt{s_{\min}\left(Q_{\boldsymbol{\lambda}}\left(\mathbb{A}\right)\right)} \tag{5.3}$$

where

$$Q_{\boldsymbol{\lambda}}\left(\mathbb{A}\right) = \sum_j \left(A_j - \lambda_j\right)^2. \tag{5.4}$$

This can be defined also in the case where the A_j are Hermitian elements in a unital C^*-algebra.

We can also select a bump function $g : \mathbb{R} \to \mathbb{R}$ with $0 \le g \le 1$ and $g(0) = 1$, some manner of a windowing function. Lin [21] looks at

$$1 - \|W(\mathbb{A})\|$$

where

$$W(\mathbb{A}) = g\left(A_1 - \lambda_1\right) g\left(A_2 - \lambda_2\right) \cdots g\left(A_d - \lambda_d\right).$$

Notice there needs to be some choice of the order in the product. This is related to the windowed LDOS [24] that looks at the trace of $\left(W(\mathbb{A})\right)^* W(\mathbb{A})$. It can be difficult to compute $g\left(A_j - \lambda_j\right)$, but it should be possible to numerically compute this in settings where some of the matrices are diagonal. For theoretical calculations, this can be a practical form of pseudospectrum [21].

Pure mathematicians have been more interested in defining just a joint spectrum from noncommuting operators rather than a joint pseudospectrum. One goal of these theories is often to develop a noncommutative functional calculus and an associated spectral mapping theorem [11,17,34]. The Clifford spectrum fails in this regard. Even a simple rescaling of matrices is too much. For applications in topological physics, the complicated way the Clifford spectrum varies as we rescale position observables while fixing the Hamiltonian is essential, as without it the spectral localizer probably fails to detect any K-theory. This is discussed in [4, §3]. See also Figure 8 in [5]. We shall see, in Theorem 5.5, that the quadratic joint spectrum does a little better at having a spectral mapping theorem.

5.2. Relation between the Clifford and quadratic pseudospectra

It is expected that all variations on multivariable operator pseudospectrum will be related somehow. Indeed, they should all be equal in the commutative case, and be close when the A_j have small commutators. This is known in one case, in that the Clifford and quadratic are of bounded distance apart. That is, we know [6] (see Equation (3.2)) that

$$\left|\left(\mu_{\boldsymbol{\lambda}}^{C}(\mathbb{A})\right)^2-\left(\mu_{\boldsymbol{\lambda}}^{Q}(\mathbb{A})\right)^2\right|\le\sum_{j<k}\|[A_j,A_k]\|. \tag{5.5}$$

It is becoming clear that one cannot make do with a single form of pseudospectrum when studying physical systems. The Clifford pseudospectrum is related to K-theory and is finding applications in photonics [3,8], acoustics [10], aperiodic structures [14], nonlinear systems [35], and even non-Hermitian systems [7,9,13,22,27]. On the other hand, showing that points in the Clifford spectrum correspond to states approximately localized in energy and position seems to require using something like Equation (5.5).

We can define the *quadratic spectrum* as

$$\Lambda^{\mathrm{Q}}(\mathbb{A})=\left\{\boldsymbol{\lambda}\in\mathbb{R}^d\middle|\,\mu_{\boldsymbol{\lambda}}^{\mathrm{Q}}(\mathbb{A})=0\right\}. \tag{5.6}$$

For Hermitian matrices A_j this equals the set of $\boldsymbol{\lambda}$ for which there is a joint eigenvector $\mathbf{v}$, nonzero with $A_j\mathbf{v}=\lambda_j\mathbf{v}$. In many cases the quadratic spectrum is empty, and we shall see it is always a subset of the Clifford spectrum. Equation (5.5) tells us that if the A_j almost commute then the two pseudospectra are close. However, close functions can have null-sets that are very different.

Proposition 5.1. *If each A_j is a bounded linear operator on Hilbert space, then $\boldsymbol{\lambda}$ is in the quadratic spectrum of $\mathbb{A}$ if and only if there is a joint asymptotic eigenvector $\{\mathbf{v}_n\}$ for $\mathbb{A}$, meaning a sequence of unit vectors with*

$$(A_j\mathbf{v}_n-\lambda_j\mathbf{v}_n)\to\mathbf{0}.$$

Proof. To see this, one must interpret the infimum in the definition $s_{\min}$ in the equation

$$\left(s_{\min}\left(Q_{\boldsymbol{\lambda}}\left(\mathbb{A}\right)\right)\right)^{\frac{1}{2}}=s_{\min}\left(\begin{bmatrix}A_1-\lambda_1 I\\ \vdots\\ A_d-\lambda_d I\end{bmatrix}\right). \quad \square$$

Lemma 5.2. *For all $d\ge 1$ there is a Clifford representation $\Gamma_1,\ldots,\Gamma_d$ and a unit vector $\mathbf{u}$ so that $\Gamma_1\mathbf{u},\ldots,\Gamma_d\mathbf{u}$ is an orthonormal set.*

Proof. We can prove this by induction. For $d=1$ we can set $\Gamma_1=\mathbf{u}=[1]$. Now assume $\gamma_1,\ldots,\gamma_{d-1}$ are a Clifford representation and $\mathbf{v}$ is a unit vector with $\gamma_1\mathbf{v},\ldots,\gamma_{d-1}\mathbf{v}$ being orthonormal. Define $\Gamma_j=\gamma_j\otimes\sigma_x$ for $j<d$ and $\Gamma_d=I\otimes\sigma_z$. These matrices are again a Clifford representation. Next define $\mathbf{u}=\mathbf{q}\otimes\mathbf{e}_1$. Then $\Gamma_j\mathbf{u}=\gamma_j\mathbf{q}\otimes\mathbf{e}_2$ except for the last one, which is $\Gamma_d\mathbf{u}=\mathbf{q}\otimes\mathbf{e}_1$. These d vectors are again orthonormal. $\square$

Theorem 5.3. *For any Hermitian A_j in a unital C^*-algebra we have*

$$\mu_{\boldsymbol{\lambda}}^{\mathrm{C}}(\mathbb{A})\le\mu_{\boldsymbol{\lambda}}^{\mathrm{Q}}(\mathbb{A})$$

and, in particular,

$$\Lambda^{\mathrm{Q}}(\mathbb{A}) \subseteq \Lambda^{\mathrm{C}}(\mathbb{A}).$$

Proof. It suffices to prove this for operators on Hilbert space. For a given $\epsilon > 0$ we can find a unit vector $\mathbf{v}$ so that

$$\sum \|A_j \mathbf{v} - \lambda_j \mathbf{v}\|^2 \leq \left(\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(\mathbb{A})\right)^2 + \epsilon.$$

We can use the Γ_j from the last lemma and so can take $\mathbf{u}$ a unit vector so that $\Gamma_1\mathbf{u}, \ldots, \Gamma_d\mathbf{u}$ is an orthonormal set. Then the various $(A_j - \lambda_j)\mathbf{v} \otimes \Gamma_j \mathbf{u}$ are orthogonal, so

$$\begin{aligned} \|L_{\boldsymbol{\lambda}}(\mathbb{A})\,(\mathbf{v}\otimes\mathbf{u})\|^2 &= \left\|\sum (A_j - \lambda_j)\mathbf{v}\otimes\Gamma_j\mathbf{u}\right\|^2 \\ &= \sum \|(A_j - \lambda_j)\mathbf{v}\|^2 \\ &\leq \left(\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(\mathbb{A})\right)^2 + \epsilon. \end{aligned}$$

This implies

$$\left(s_{\min}\left(L_{\boldsymbol{\lambda}}(\mathbb{A})\right)\right)^2 \leq \left(\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(\mathbb{A})\right)^2 + \epsilon.$$

As this is true for all positive ϵ the result follows. □

5.3. Symmetry in the quadratic pseudospectrum

There are symmetry properties in the Clifford spectrum that must follow from symmetry properties in $\mathbb{A}$ [4]. Here we prove the same implication, but for the quadratic case.

Theorem 5.4. *Suppose $(A_1, \ldots, A_d)$ is a d-tuple of Hermitian elements in the unital C^*-algebra $\mathcal{A}$ and that $U \in O(d)$. Suppose $\boldsymbol{\lambda} \in \mathbb{R}^d$. The d elements*

$$\hat{A}_j = \sum_s u_{js} A_s$$

are also Hermitian, and

$$\mu^Q_{U\boldsymbol{\lambda}}\left(\hat{A}_1, \ldots, \hat{A}_d\right) = \mu^Q_{\boldsymbol{\lambda}}\left(A_1, \ldots, A_d\right)$$

and

$$\mu^C_{U\boldsymbol{\lambda}}\left(\hat{A}_1, \ldots, \hat{A}_d\right) = \mu^C_{\boldsymbol{\lambda}}\left(A_1, \ldots, A_d\right).$$

Proof. The elements $\hat{A}_j$ are Hermitian since the u_{jk} are all real. The statement regarding the Clifford pseudospectrum follows from the proof of [4, Theorem 2.1].

We find

$$\begin{aligned} \sum_j \hat{A}_j^2 &= \sum_r \sum_s \sum_j u_{jr} u_{js} A_r A_s \\ &= \sum_r A_r^2 \\ &= \sum_j A_j^2. \end{aligned}$$

Setting

$$\tilde{\lambda}_j = \sum_s u_{js}\lambda_s$$

we next compute

$$\begin{aligned}\sum_j \tilde{\lambda}_j \hat{A}_j &= \sum_r \sum_s \sum_j u_{js}u_{jr}\lambda_s A_r \\ &= \sum_r \lambda_r A_r \\ &= \sum_j \lambda_j A_j .\end{aligned}$$

Finally

$$\begin{aligned}\sum_j \tilde{\lambda}_j^2 &= \sum_r \sum_s \sum_j u_{rj}u_{sj}\lambda_r\lambda_s \\ &= \sum_r \lambda_r^2 .\end{aligned}$$

Thus

$$\begin{aligned}Q_{\tilde{\boldsymbol{\lambda}}}\left(\hat{A}_1, \ldots, \hat{A}_d\right) &= \sum_j \hat{A}_j^2 - 2\sum_j \tilde{\lambda}_j \hat{A}_j + \sum_j \tilde{\lambda}_j^2 I \\ &= \sum_j A_j^2 - 2\sum_j \lambda_j A_j + \sum_j \lambda_j I \\ &= Q_{\boldsymbol{\lambda}}\left(A_1, \ldots, A_d\right) . \quad \square\end{aligned}$$

The spectral mapping theorem fails for the Clifford spectrum outside of [6], the analog of Theorem 5.4. Indeed, even for linear transformations applied to three 2-by-2 matrices, there are examples where the Clifford spectrum does not follow the linear transformation [12, §4]. Next we show that the quadratic spectrum has a less-limited form of spectral mapping theorem. We are very grateful to the anonymous referee for pointing this out.

Theorem 5.5. *Suppose $(A_1, \ldots, A_d)$ is a d-tuple of Hermitian elements in the unital C^*-algebra $\mathcal{A}$ and that $W \in \mathrm{GL}_d(\mathbb{R})$. Suppose $\boldsymbol{\lambda} \in \mathbb{R}^d$. The d elements*

$$\hat{A}_j = \sum_s w_{js} A_s$$

are also Hermitian and

$$\Lambda^Q(\hat{\mathbb{A}}) = W\Lambda^Q(\mathbb{A})$$

and

$$s_{\min}(W)\mu^Q_{\boldsymbol{\lambda}}(\mathbb{A}) \le \mu^Q_{W\boldsymbol{\lambda}}(\hat{\mathbb{A}}) \le \|W\|\mu^Q_{\boldsymbol{\lambda}}(\mathbb{A}).$$

Proof. For this proof we use the description of the quadratic pseudospectrum in term of joint eigenvectors. Let $\hat{\boldsymbol{\lambda}} = W\boldsymbol{\lambda}$. If there is unit vector $\mathbf{v}$ with

$$A_j\mathbf{v} = \lambda_j\mathbf{v}$$

for all j then

$$\begin{aligned}\hat{A}_j\mathbf{v} &= \sum_s w_{js}A_s\mathbf{v} \\ &= \sum_s w_{js}\lambda_s\mathbf{v} \\ &= \hat{\lambda}_s\mathbf{v}.\end{aligned}$$

This proves the inclusion $\Lambda^Q(\hat{\mathbb{A}}) \supseteq W\Lambda^Q(\mathbb{A})$. The reverse inclusion follows if we replace W by W^{-1}.

For the quadratic pseudospectrum, we can apply the singular value composition of W and Lemma 2.4 to reduce to the case where W is diagonal, with positive diagonal entries $s_1 \leq s_2 \leq \cdots \leq s_d$. Then

$$\begin{aligned}\sum_j \left\|\hat{A}_j\mathbf{v} - \hat{\lambda}_j\mathbf{v}\right\|^2 &= \sum_j \|s_jA_j\mathbf{v} - s_j\lambda_j\mathbf{v}\|^2 \\ &= \sum_j s_j^2\|A_j\mathbf{v} - \lambda_j\mathbf{v}\|^2 \\ &\leq s_d^2\sum_j \|A_j\mathbf{v} - \lambda_j\mathbf{v}\|^2\end{aligned}$$

which proves

$$\mu^Q_{W\boldsymbol{\lambda}}\left(\hat{\mathbb{A}}\right) \leq \|W\|\mu^Q_{\boldsymbol{\lambda}}(\mathbb{A}).$$

Applying this to W^{-1} and we find

$$\mu^Q_{W\boldsymbol{\lambda}}\left(\hat{\mathbb{A}}\right) \geq s_{\min}(W)\mu^Q_{\boldsymbol{\lambda}}(\mathbb{A}). \quad \square$$

If Q is a unitary and we let $\mathbb{B} = (QA_1Q^*, \ldots, QA_1Q^*)$ then $L_{\boldsymbol{\lambda}}(\mathbb{B})$ is unitarily equivalent to $L_{\boldsymbol{\lambda}}(\mathbb{A})$ and $Q_{\boldsymbol{\lambda}}(\mathbb{B})$ is unitarily equivalent to $Q_{\boldsymbol{\lambda}}(\mathbb{A})$. Thus we can use Theorem 5.4 to obtain the following theorem.

Theorem 5.6. *Suppose $(A_1, \ldots, A_d)$ are Hermitian elements in the unital C^*-algebra $\mathcal{A}$ and that $U \in O(d)$. Let*

$$\hat{A}_j = \sum_s u_{js}A_s.$$

If there exists a unitary Q in $\mathcal{A}$ such that $Q\hat{A}_jQ^* = A_j$ for all j then

$$\mu^{\mathrm{C}}_{U\boldsymbol{\lambda}}(A_1, \ldots, A_d) = \mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(A_1, \ldots, A_d)$$

and

$$\mu^{\mathrm{Q}}_{U\boldsymbol{\lambda}}(A_1, \ldots, A_d) = \mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(A_1, \ldots, A_d).$$

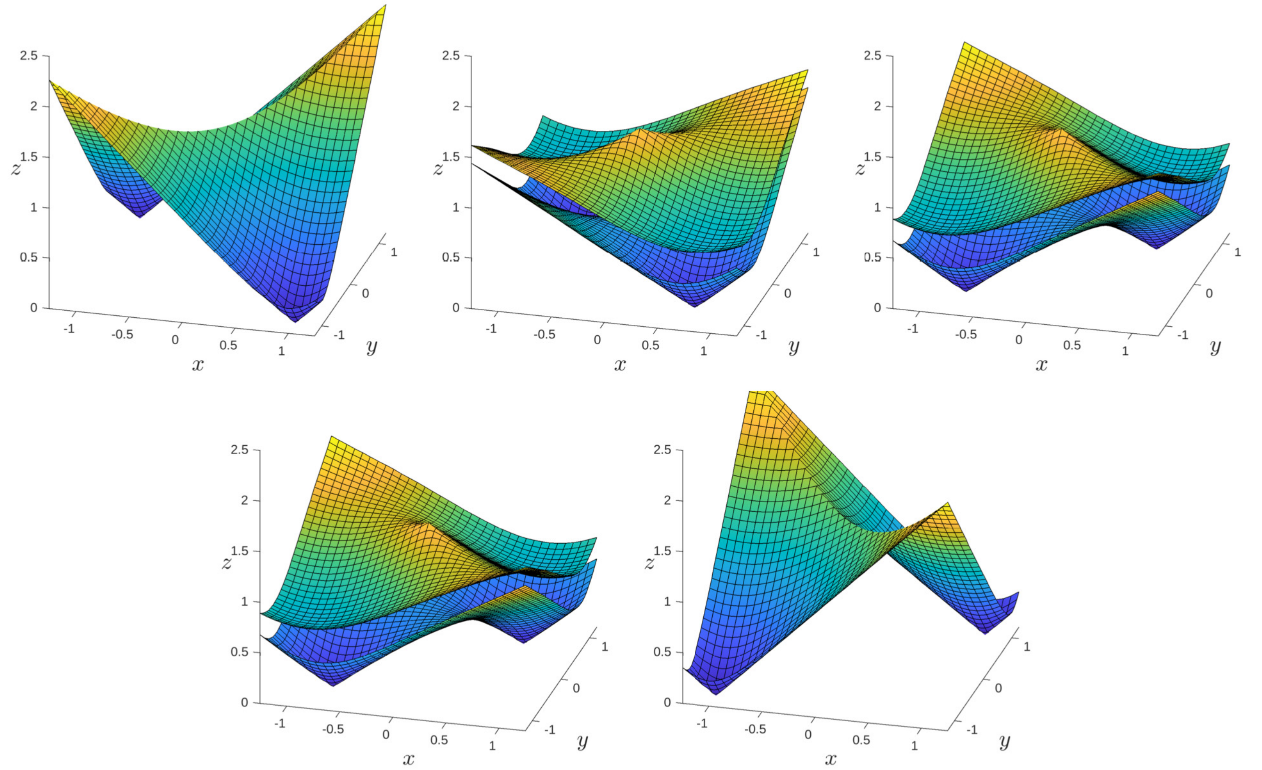

Fig. 6.1. Plotted together the Clifford and quadratic pseudospectrum for (U, V_z), as defined in (6.3),) for various values of z. From the top-left to the bottom-right, the values used are $z = 1$, $z = 0.6$, $z = 0$, $z = -0.5$, $z = -1$.

6. The C^*-algebra generated by two projections

The universal unital C^*-algebra for the relations that define two orthogonal projections [29] is

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_2) = \{f \in C([-1,1], \boldsymbol{M}_2) \mid f(-1) \text{ and } f(1) \text{ are diagonal}\}. \tag{6.1}$$

We will now compute both forms of pseudospectrum for this universal pair of projections. The formulas derived will come out simpler if we instead think in terms of a universal pair of unitary operators of order two. We might see these as two incompatible dichotomous observables.

The generators of $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ are U and V where

$$U(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; V(z) = \begin{bmatrix} z & \sqrt{1-z^2} \\ \sqrt{1-z^2} & -z \end{bmatrix}. \tag{6.2}$$

First we work on the representation corresponding to evaluation at a fixed z. Here the Hilbert space is only two-dimensional and the calculations are not too extensive. They are a bit tedious so we utilized a computer algebra package here.

Lemma 6.1. *Suppose* $-1 \leq z \leq 1$. *If*

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; V_z = \begin{bmatrix} z & \sqrt{1-z^2} \\ \sqrt{1-z^2} & -z \end{bmatrix} \tag{6.3}$$

then

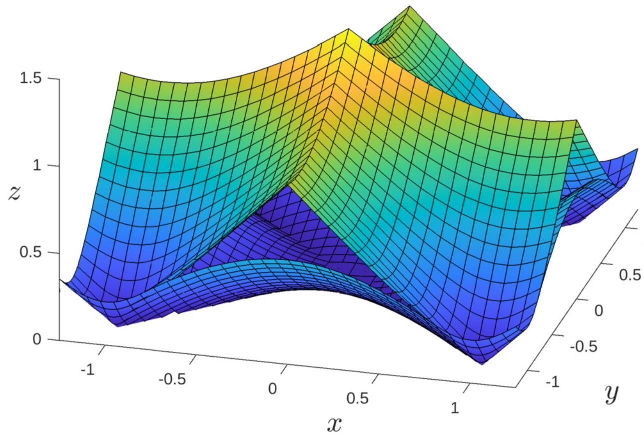

Fig. 6.2. The Clifford and quadratic pseudospectrum for the universal pair of order-two unitary matrices (U, V) are plotted together here.

$$\mu^{\mathrm{Q}}_{(x,y)}(U, V_z) = \sqrt{x^2 + y^2 + 2 - 2\sqrt{x^2 + 2zxy + y^2}}$$

and

$$\mu^{\mathrm{C}}_{(x,y)}(U, V_z) = \sqrt{x^2 + y^2 + 2 - 2\sqrt{x^2 + 2xyz + y^2 + 1 - z^2}}.$$

Proof. We compute the smallest eigenvalue of

$$Q_{(x,y)}(U, V_z) = (U - xI)^2 + (V_z - yI)^2,$$

finding that this equals

$$Q_{(x,y)} = \begin{bmatrix} x^2 + y^2 - 2x - 2zy + 2 & -2y\sqrt{1 - z^2} \\ -2y\sqrt{1 - z^2} & x^2 + y^2 + 2x + 2zy + 2 \end{bmatrix}.$$

This has eigenvalues

$$x^2 + y^2 + 2 \pm 2\sqrt{x^2 + 2zxy + y^2}.$$

The smaller of the eigenvalues is with the minus sign. To get to $\mu^{Q}_{(x,y)}(P, Q_z)$ we apply square root.

For the Clifford pseudospectrum, we need the eigenvalues of

$$\begin{bmatrix} 0 & (U - xI) - i(V_z - yI) \\ (U - xI) + i(V_z - yI) & 0 \end{bmatrix}$$

which are determined by the singular values of

$$(U - xI) + i(V_z - yI).$$

These are

$$\sqrt{x^2 + y^2 + 2 \pm 2\sqrt{x^2 + 2xyz + y^2 + 1 - z^2}}. \quad \square$$

The pseudospectra of U, V_z for a few values of z are plotted in Fig. 6.1. The pseudospectra of the unversal U, V is shown in Fig. 6.2.

Theorem 6.2. *For the universal pair of unitary operators of order two, so U and V as in (6.2) in the universal C^*-algebra (6.1), the quadratic pseudospectrum is*

$$\mu^{\mathrm{Q}}_{(x,y)}(U,V) = \operatorname{dist}\Big((x,y), \Big\{(-1,-1),(-1,1),(1,-1),(1,1)\Big\}\Big)$$

and the Clifford pseudospectrum is

$$\mu^{\mathrm{C}}_{(x,y)}(U,V) = \sqrt{x^2+y^2+2-2\sqrt{x^2y^2+x^2+y^2+1}} \tag{6.4}$$

when $-1 \le xy \le 1$, and otherwise

$$\mu^{\mathrm{C}}_{(x,y)}(U,V) = \mu^{\mathrm{Q}}_{(x,y)}(U,V).$$

In particular, the quadratic spectrum is the set of the four points $(\pm 1, \pm 1)$ and the Clifford spectrum is a cross, the union of the line segment from $(-1,-1)$ to $(1,1)$ and the line segment from $(-1,1)$ to $(1,-1)$.

Proof. First the quadratic case. For each (x,y) we need to minimize

$$\mu^{\mathrm{Q}}_{(x,y)}(U,V_z) = \sqrt{x^2+y^2+2-2\sqrt{x^2+2zxy+y^2}}$$

over the range $-1 \le z \le 1$. This is constant when $x=0$ or $y=0$ and in the remaining cases, is maximized when $z=-1$ or when $z=1$. Since

$$x^2 \pm 2xy + y^2 = (x \pm y)^2$$

we find that

$$\mu^{\mathrm{Q}}_{(x,y)}(U,V_1) = \sqrt{x^2+y^2+2-2\sqrt{x^2+2xy+y^2}} = \begin{cases} \sqrt{(x+1)^2+(y+1)^2} & \text{if } x+y \le 0 \\ \sqrt{(x-1)^2+(y-1)^2} & \text{if } x+y \ge 0 \end{cases}.$$

This means $\mu^{\mathrm{Q}}_{(x,y)}(U,V_1)$ is the distance of (x,y) to the set $\{(-1,-1),(1,1)\}$. Similarly $\mu^{\mathrm{Q}}_{(x,y)}(U,V_{-1})$ is the distance of (x,y) to the set $\{(1,-1),(-1,1)\}$.

Now we look to the Clifford case. For each (x,y) we need to minimize

$$\mu^{\mathrm{C}}_{(x,y)}(U,V_z) = \sqrt{x^2+y^2+2-2\sqrt{x^2+2xyz+y^2+1-z^2}}. \tag{6.5}$$

This will happen at the maximum of

$$g(z) = x^2+2xyz+y^2+1-z^2.$$

This has derivative

$$g'(z) = 2xy - 2z$$

so the maximum value of g seems like it will be at $z=z_0$ where $z_0 = xy$ but we need to ensure this is with $-1 \le z \le 1$. If (x,y) is in the region specified by $-1 \le xy \le 1$, a region that includes $[-1,1]\times[-1,1]$, then the value of $\mu^{\mathrm{C}}_{(x,y)}(U,V_z)$ is found at z_0. Substituting z_0 for z in Eq. (6.5) yields Eq. (6.4). □

7. A hemisphere as Clifford spectrum

Example 3.2 tells us we can find essentially any finite-dimensional compact space as the Clifford spectrum of some set of commuting Hermitian operators. In the finite-dimensional case we need to introduce noncommutativity to get Clifford spectrum in dimension greater than zero. Many papers have examined the spaces that one can get from finite matrices, including [1,4,12,31]. With three matrices it is possible to have as Clifford spectrum a 2-manifold or a surface with cusps. With four matrices one can get 2-manifolds and 3-manifold, for example. The simplest example leading to a closed surface is the Pauli matrices themselves [19], where

$$\Lambda^{\mathrm{C}}(\sigma_x, \sigma_y, \sigma_z) = S^2.$$

We find some odd behavior when we modify this example to be infinite-dimensional.

Consider

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \ A_2 = \frac{1}{2}\begin{bmatrix} 0 & -i & & & \\ i & 0 & -i & & \\ & i & 0 & -i & \\ & & i & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}, \ A_3 = \begin{bmatrix} b & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}. \tag{7.1}$$

If we set $b = 1/2$ we have operators that are somewhat like $\frac{1}{2}\sigma_x$, $\frac{1}{2}\sigma_y$, $\frac{1}{2}\sigma_z$. We will find that for this class of examples, we get a distorted hemisphere that bifurcates for large b. Indeed, when b is large the spectrum of A_3 dominates. This is in some way a manifestation of the fact that the K-theory of $\mathcal{B}(\mathcal{H})$ is trivial so there can be no local index stabilizing the Clifford spectrum.

Notice that A_1, A_2, A_3 are compact perturbations of

$$K_1 = \frac{1}{2}(S^* + S),\ K_2 = \frac{i}{2}(S^* - S),\ K_3 = 0$$

where S is the backwards shift operator. Thus the essential Clifford spectrum in this example will not depend on b.

Proposition 7.1. *For any $b \geq 0$, if A_1, A_2, A_3 are defined according to Equation (7.1), then*

$$\Lambda^{\mathrm{Q}}(A_1, A_2, A_3) = \mathbb{T} \times \{0\}.$$

Proof. We need to look for joint asymptotic eigenvectors for A_1, A_2 and A_3, following Proposition 5.1. Equivalently, we a looking for a joint asymptotic eigenvector $\{\mathbf{v}_n\}$ for A_3 and both the forward shift $S^* = A_1 - iA_2$ the backward shift $S = A_1 + iA_2$. Since A_3 has finite spectrum, one can assume without loss of generality that all of the $\mathbf{v}_n$ are in one of the eigenspaces for A_3 The approximate point spectrum of the forward shift is $\mathbb{T}$ so we have proven that the quadratic spectrum is contained in $\mathbb{T} \times \{0, b\}$. If $b > 0$ then the eigenspace for b is one-dimensional and cannot produce an asymptotic eigenvector for the forward shift. Therefore the quadratic spectrum must be a subset of $\mathbb{T} \times \{0\}$.

We can see the other inclusion explicitly. Given $|\lambda| = 1$ we can define

$$\mathbf{v}_n = (1/\sqrt{n})(0, \lambda^1, \lambda^2, \ldots, \lambda^n, 0, 0, 0, \ldots).$$

Each $\mathbf{v}_n$ is a null-vector for A_3 and collectively these form a joint asymptotic eigenvalue for S and S^* for λ and $\overline{\lambda}$, respectively. This tells us that that $(\Re(z), \Im(z), 0)$ is in the quadratic spectrum of A_1, A_2, A_3. $\square$

Proposition 7.2. *For any $b \geq 0$, if A_1, A_2, A_3 are defined according to Equation (7.1), then*

$$\Lambda^{\mathrm{C}}_{\mathrm{e}}(A_1, A_2, A_3) = \mathbb{T} \times \{0\}.$$

Proof. Since the essential spectrum of the unilateral shift is the unit circle we know that

$$\Lambda^{\mathrm{C}}_{\mathrm{e}}(A_1, A_2) = \mathbb{T}.$$

Now Theorem 3.4 implies

$$\Lambda^{\mathrm{C}}_{\mathrm{e}}(A_1, A_2, A_3) = \Lambda^{\mathrm{C}}_{\mathrm{e}}(A_1, A_2) \times \{0\}$$

so we are done. □

Example 7.3. For any $0 \leq b \leq 2.25$, if A_1, A_2, A_3 are defined according to Equation (7.1), then

$$\Lambda^{\mathrm{C}}(A_1, A_2, A_3).$$

equals the set of points (x, y, z) that satisfy the equation

$$-bz^4 + 3b^2z^3 - (3b^3 + b)z^2 + \left(-b^2r^2 + 2b^2 + b^4 + r^2\right) z + b^3r^2 - b^3 + br^4 - br^2 = 0, \tag{7.2}$$

subject to the constraints $0 \leq z \leq b$ and $r^2 \leq 1$, where $r^2 = x^2 + y^2$. When $b = 1$ this simplifies to

$$-z^4 + 3z^3 - 4z^2 + 3z - 1 + r^4 = 0.$$

When $b = 0$ this can be easily verified using Theorem 3.4. We provide a rigorous proof also for the special case $b = 1$. For the general case, we take an experimental mathematics approach and provide evidence that the Clifford spectrum is likely as claimed. We use computer algebra to derive an equality and an inequality that together determine part of the Clifford spectrum and then estimate the joint solution numerically.

The cutoff at $b = 2.25$ was selected to be just past the bifurcation that occurs at $b = 2$. A supplemental video illustrates this bifurcation, while Fig. 7.1 contains three of the frames of that video. For large b the spectrum of A_3 becomes dominant and so we do not expect anything interesting to show up past $b = 2.25$.

For any $b > 0$ it is easy to see that $e^{i\theta}S$ is unitarily equivalent to S. This implies that

$$(\cos(\theta)A_1 + \sin(\theta)A_2,\ \cos(\theta)A_1 - \sin(\theta)A_2,\ A_3)$$

is unitarily equivalent to A_1, A_2, A_3. We can now apply Theorem 2.5 to conclude that $\Lambda(A_1, A_2, A_3)$ has rotational symmetry in the x-y plane. Thus, we can restrict our attention to the localizer with $\boldsymbol{\lambda} = (x, 0, z)$ for $x \geq 0$.

The localizer naturally lives in $\mathbf{M}_2(\mathcal{B}(\mathcal{H}))$ but we can identify that with $\mathcal{B}(\mathcal{H})$ via the usual shuffling of basis elements. Making this identification we can compute $L_{(x,0,z)}$ as follows. Let E_{ij} denote the usual two-by-two matrix units, so that $\sigma_y = -iE_{12} + iE_{21}$ etc. Since

$$\begin{aligned} A_1 \otimes \sigma_x + A_2 \otimes \sigma_y &= A_1 \otimes (E_{12} + E_{21}) + iA_2 \otimes (-E_{12} + E_{21}) \\ &= (A_1 - iA_2) \otimes E_{12} + (A_1 + iA_2) \otimes E_{21} \\ &= S^* \otimes E_{12} + S \otimes E_{21} \end{aligned}$$

we find

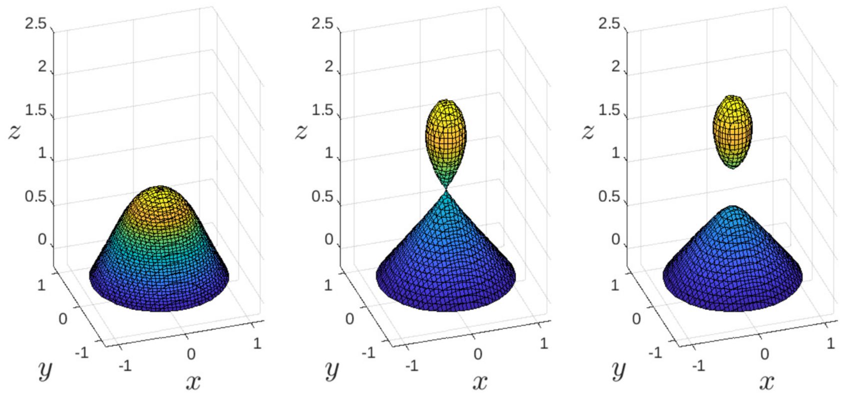

Fig. 7.1. The Clifford spectrum for the three operators in Equation (7.1). $b = 1.00$ (left); $b = 2.00$ (center); $b = 2.05$ (right). A supplemental video shows these plots for additional values of b.

$$A_1 \otimes \sigma_x + A_2 \otimes \sigma_y = \begin{bmatrix} 0 & 0 & & & & & \\ 0 & 0 & 1 & & & & \\ & 1 & 0 & 0 & & & \\ & & 0 & 0 & 1 & & \\ & & & 1 & 0 & 0 & \\ & & & & 0 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

Therefore

$$L_{(x,0,z)} = (A_1 - xI) \otimes \sigma_x + A_2 \otimes \sigma_y + (A_3 - zI) \otimes \sigma_z$$

works out as

$$L_{(x,0,z)} = \begin{bmatrix} b-z & -x & & & & & \\ -x & z-b & 1 & & & & \\ & 1 & -z & -x & & & \\ & & -x & z & 1 & & \\ & & & 1 & -z & -x & \\ & & & & -x & z & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

We already worked out the essential Clifford spectrum so we already know that 0 is in the essential spectrum of the spectral localizer. When $z = 0$, Equation (7.2) becomes $r^4 - 1 = 0$ so we see immediately that that points in the essential spectrum $\mathbb{T} \times \{0\}$ always satisfy this equation. What is left is to find the discrete spectrum, that is, those points $\boldsymbol{\lambda} = (x, 0, z)$ such that $L_{\boldsymbol{\lambda}}(\mathbb{A})$ is Fredholm but not invertible. At such points $L_{\boldsymbol{\lambda}}(\mathbb{A})$ will have a nontrivial kernel, because the spectral localizer is self-adjoint. Thus, we search for values of x and z that lead to null vectors for $L_{(x,0,z)}$.

The calculations in the general case are complicated so now it is time to restrict to the case of $b = 1$.

We need to discuss a special case. First we deal with $x = 0$. This makes $L_{(x,0,z)}$ block diagonal, with blocks

$$[1-z], \begin{bmatrix} z-1 & 1 \\ 1 & -z \end{bmatrix}, \begin{bmatrix} z & 1 \\ 1 & -z \end{bmatrix}$$

that have determinants $1 - z$, $-z^2 + z - 1$ and $-z^2 - 1$, so only $z = 1$ leads to $L_{(x,0,z)}$ being singular.

Now assume $x \geq 0$. Let us assume that there exists a nonzero vector $\boldsymbol{a} = (a_1, a_2, \dots) \in l^2(\mathbb{N})$ so that $L_{(x,0,z)}\boldsymbol{a} = 0$. We can rescale $\boldsymbol{a}$ so that $a_1 = 1$. This means

$$\begin{aligned} (1 - z) - xa_2 &= 0, \\ -x + (z - 1)a_2 + a_3 &= 0, \end{aligned}$$

and for $n \geq 2$,

$$\begin{aligned} a_{2n-2} - za_{2n-1} - xa_{2n} &= 0, \\ -xa_{2n-1} + za_{2n} + a_{2n+1} &= 0. \end{aligned}$$

Thus, for $n \geq 2$,

$$a_{2n} = -\frac{z}{x}a_{2n-1} + \frac{1}{x}a_{2n-2}$$

and

$$\begin{aligned} a_{2n+1} &= -za_{2n} + xa_{2n-1} \\ &= -z\left(-\frac{z}{x}a_{2n-1} + \frac{1}{x}a_{2n-2}\right) + xa_{2n-1} \\ &= \left(\frac{z^2}{x} + x\right)a_{2n-1} - \frac{z}{x}a_{2n-2}. \end{aligned}$$

If we set

$$M = \begin{bmatrix} \frac{1}{x} & -\frac{z}{x} \\ -\frac{z}{x} & \frac{x^2+z^2}{x} \end{bmatrix}$$

the relation on $\boldsymbol{a}$ is

$$\begin{bmatrix} a_{2n} \\ a_{2n+1} \end{bmatrix} = M \begin{bmatrix} a_{2n-2} \\ a_{2n-1} \end{bmatrix}.$$

We also know

$$a_2 = \frac{1-z}{x}$$

and

$$\begin{aligned} a_3 &= x + (1 - z)a_2 \\ &= x + (1 - z)\frac{1-z}{x} \end{aligned}$$

so have an initial condition

$$\boldsymbol{v}_0 = \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{1}{x}\begin{bmatrix} -z + 1 \\ z^2 - 2z + x^2 + 1 \end{bmatrix}.$$

Now let's pause to take care of the case of $z = 1$ and $x > 0$. In that situation, we have

$$M = \begin{bmatrix} \frac{1}{x} & -\frac{1}{x} \\ -\frac{1}{x} & \frac{x^2+1}{x} \end{bmatrix}$$

so that

$$\boldsymbol{v}_0 = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

is not going to be an eigenvector. Thus, the only Clifford spectrum on the line $z = 1$ is the point $(x, 0, z) = (0, 0, 1)$ that we found above.

For the general case, we need to know about the eigenvalues of the real symmetric matrix M and see that

$$\operatorname{Tr}(M) = \frac{x^2 + z^2 + 1}{x}$$

and $\det(M) = 1$ we find that the eigenvalues of the real symmetric matrix M are

$$\frac{1}{2}\left(\operatorname{Tr}(M) \pm \sqrt{(\operatorname{Tr}(M))^2 - 4}\right).$$

We get a double real eigenvalue when

$$\frac{x^2 + z^2 + 1}{x} = 2$$

but this leads to $x = 1$ and $z = 0$ which we have excluded. We get two complex eigenvalues when

$$\frac{x^2 + z^2 + 1}{x} < 2$$

which is equivalent to

$$(x - 1)^2 + z^2 < 0$$

and that cannot happen. What we need look at then is where M has two positive eigenvalues, one outside of $(0, 1)$ and one inside. The only way to have $\boldsymbol{a}$ square-summable is if the initial vector lands in the eigenspace for the eigenvalue closer to 0.

Let us find out when $M\boldsymbol{v}_0$ is parallel to $\boldsymbol{v}_0$. Since

$$M\boldsymbol{v}_0 = \frac{1}{x^2}\begin{bmatrix} -z^3 + 2z^2 - (x^2 + 2)z + 1 \\ z^4 - 2z^3 + (2x^2 + 2)z^2 - (2x^2 + 1)z + x^4 + x^2 \end{bmatrix}$$

we need to solve

$$\frac{-z^3 + 2z^2 - (x^2 + 2)z + 1}{-z + 1} = \frac{z^4 - 2z^3 + (2x^2 + 2)z^2 - (2x^2 + 1)z + x^4 + x^2}{z^2 - 2z + x^2 + 1}.$$

For $z \neq 1$ this is equivalent to

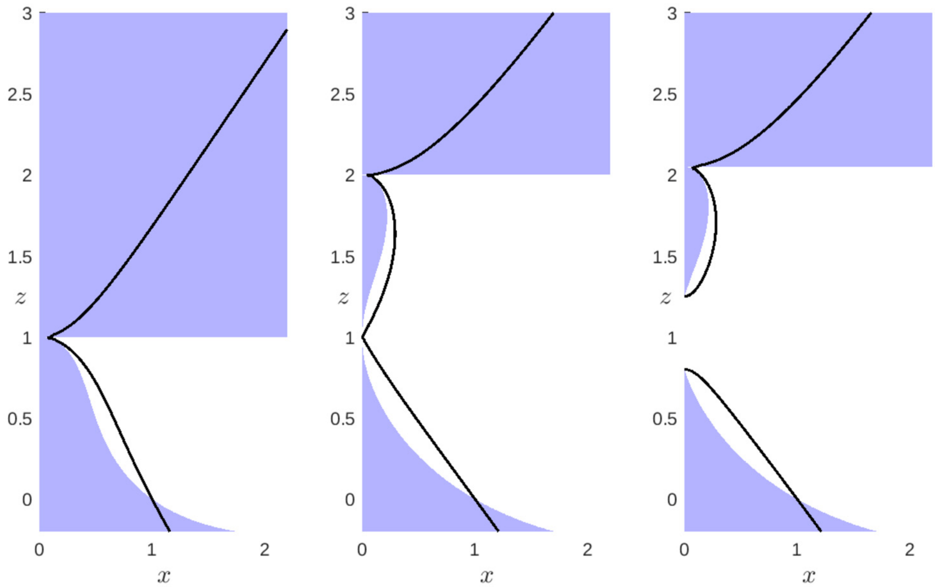

Fig. 7.2. Plots of $e(x,z)=0$ (dark curves) and $f(x,z)\geq 0$ (light areas, switched to $\leq$ above $z=b$) for $b=1.00$, $b=2.00$, $b=2.05$. A supplemental video shows these plots for additional values of b.

$$\frac{-z^4+3z^3-4z^2+3z+x^4-1}{(z-1)(x^2+(z-1)^2)}=0$$

so we get the curve $e(x,z)=0$ where

$$e(x,z)=-z^4+3z^3-4z^2+3z+x^4-1.$$

What about the associated eigenvalue α? We need to be less than 1. First, we deal with the case where $z<1$. Here the first coordinate of the initial vector $\boldsymbol{v}_0$ will be positive so $\alpha<1$ will happen when

$$\frac{-z+1}{x}>\frac{-z^3+2z^2-(x^2+2)z+1}{x^2}.$$

As x^2 is positive this is equivalent to

$$f(x,z)=z^3-2z^2+(x^2-x+2)z+x-1>0.$$

For $z>1$ we need to know when $f(x,z)$ is negative.

We find

$$\frac{\partial f}{\partial z}=3z^2-4z+x^2-x+2=(x-\tfrac{1}{2})^2+3(z-\tfrac{2}{3})^2+\tfrac{5}{12}$$

is always positive. The plots of the zero-locus of e and f are shown in Fig. 7.2. These seem to cross at $(x,z)=(0,1)$ and $(x,z)=(1,0)$ and the curve for $f=0$ is below the curve for $e=0$ between those two crossings. If we believe the computer-generated plots then we can conclude that the discrete Clifford spectrum is on the curve $e(x,z)=0$ restricted to the region $0\leq x,z\leq 1$.

We can verify the above claim rigorously, with some uninteresting calculus. We put the rest of the proof in Appendix A.

When $b\neq 1$ the polynomials $e(x,z)$ and $f(x,z)$ are trickier. As such, we resorted to using symbolic computer algebra for various values of b and numerically plotting the curve $e(x,z)=0$ and regions associated to $f(x,z)$. While we do not have a fully rigorous proof, we feel this is sufficient to understand this example.

8. Momentum and position – an unbounded example

We hope to push the idea of joint pseudospectrum into the realm of unbounded operators. Indeed, the spectral localizer has been applied to unbounded operators in [30,33] for example. However, the goal has typically been to understand a K-theory operation and not to compute the Clifford spectrum. Here we consider a classic example, and do not attempt a general theory. The example is a classic interpretation of position and momentum.

We will work on the Hilbert space $L^2(\mathbb{R})$, and in particular on the subspace $\mathcal{S}$ of Schwartz functions, where we consider only f for which $x^m D^n f(x)$ is always bounded. Here we use

$$D(f)(x) = \frac{d}{dx}(f(x))$$

to denote the standard differential operator. We consider two unbounded operators, $P, Q : \mathcal{S} \to \mathcal{S}$ where $P = -iD$ and $Q(f)(x) = xf(x)$. These are both symmetric, but not self-adjoint.

We can consider

$$Q_{\boldsymbol{\lambda}}(P,Q) = (P - \lambda_1 I)^2 + (Q - \lambda_2 I)^2 : \mathcal{S} \to \mathcal{S}.$$

and define

$$\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(P,Q) = \sqrt{\min_{\|f\|=1} \|Q_{\boldsymbol{\lambda}}(P,Q)(f)\|}.$$

The Clifford pseudospectrum is also easy to define, as

$$\mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(P,Q) = \min_{\|f\|=1} \|L_{\boldsymbol{\lambda}}(P,Q)(f)\|$$

where

$$L_{\boldsymbol{\lambda}}(P,Q) = \begin{bmatrix} 0 & (P-\lambda_1 I) - i(Q-\lambda_2 I) \\ (P-\lambda_1 I) + i(Q-\lambda_2 I) & 0 \end{bmatrix}$$

is taken to be an operator on $S \oplus \mathcal{S}$. In [12], this example of Clifford pseudospectrum was computed, with

$$\mu^{\mathrm{C}}_{\boldsymbol{\lambda}}(P,Q) \equiv 0$$

being the result.

We will need some alternate characterizations of the quadratic pseudospectrum of P and Q, as proven in [6] for the matrix case. We can consider

$$M_{\boldsymbol{\lambda}}(P,Q) = \begin{bmatrix} P - \lambda_1 I \\ Q - \lambda_2 I \end{bmatrix} : \mathcal{S} \to \mathcal{S} \oplus \mathcal{S}.$$

and can check that

$$\begin{aligned} \mu^{Q}_{\boldsymbol{\lambda}}(P,Q) &= \sqrt{\min_{\|f\|=1} \left\| (M_{\boldsymbol{\lambda}}(P,Q)(f))^* \, (M_{\boldsymbol{\lambda}}(P,Q)(f)) \right\|} \\ &= \min_{\|f\|=1} \|M_{\boldsymbol{\lambda}}(P,Q)(f)\| \\ &= \min_{\|f\|=1} \sqrt{\|Pf - \lambda_1 f\|^2 + \|Qf - \lambda_2 f\|^2}. \end{aligned}$$

Since

$$\|A\boldsymbol{v}-\lambda\boldsymbol{v}\|^2=\Delta_{\boldsymbol{v}}^2 A+\left(\mathrm{E}_{\boldsymbol{v}}(A)-\lambda\right)^2$$

for any Hermitian operator, we can find lower bounds on the quadratic pseudospectrum by utilizing results on lower bounds on the sum of uncertainty. For example, there are lower bounds on $\Delta_{\boldsymbol{v}}^2 A+\Delta_{\boldsymbol{v}}^2 B$ in [25]. For simplicity, we give a simplification of a proof from [25] and directly prove the following.

Lemma 8.1. *If $\boldsymbol{v}$ is a unit vector and A and B are Hermitian operators then*

$$\|A\boldsymbol{v}\|^2+\|B\boldsymbol{v}\|^2 \geq \left|\langle i\,[A,B]\,\boldsymbol{v},\boldsymbol{v}\rangle\right|.$$

Proof. We first calculate

$$\|(A\mp iB)\,\boldsymbol{v}\|^2=\|A\boldsymbol{v}\|^2+\|B\boldsymbol{v}\|^2 \mp \langle i\,[A,B]\,\boldsymbol{v},\boldsymbol{v}\rangle$$

so we have

$$\|A\boldsymbol{v}\|^2+\|B\boldsymbol{v}\|^2=\|(A\mp iB)\,\boldsymbol{v}\|^2 \pm \langle i\,[A,B]\,\boldsymbol{v},\boldsymbol{v}\rangle\,.$$

An easy consequence is then

$$\|A\boldsymbol{v}\|^2+\|B\boldsymbol{v}\|^2 \geq \pm \langle i\,[A,B]\,\boldsymbol{v},\boldsymbol{v}\rangle\,. \quad \square$$

Theorem 8.2. *For P and Q as above,*

$$\mu_{\boldsymbol{\lambda}}^{\mathrm{Q}}(P,Q)=1$$

for all $\boldsymbol{\lambda}$.

Proof. We start by establishing that $\mu_{\boldsymbol{\lambda}}^{\mathrm{Q}}(P,Q)$ is constant. The shift operator $f(x)\mapsto f(x-\lambda_2)$ gives a unitary U that takes $\mathcal{S}$ onto $\mathcal{S}$. Since

$$\begin{aligned} U^*PU(f(x)) &= -iU^*(f'(x-\lambda_2)) \\ &= -if'(x) \\ &= P(f(x)) \end{aligned}$$

and

$$\begin{aligned} U^*QU(f(x)) &= U^*(xf(x-\lambda_2)) \\ &= (x+\lambda_2)f(x) \end{aligned}$$

we find $U^*PU=P$ and $U^*QU=Q+\lambda_2$. Therefore $Q_{(\lambda_1,\lambda_2)}(P,Q)$ is unitarily equivalent to $Q_{(\lambda_1,0)}(P,Q)$ and so

$$\mu_{(\lambda_1,\lambda_2)}^{\mathrm{Q}}(P,Q)=\mu_{(\lambda_1,0)}^{\mathrm{Q}}(P,Q).$$

Likewise, we can use $f(x)\mapsto e^{2\pi i\lambda_1 x}f(x)$ so set up a unitary that shows

$$\mu^{\mathrm{Q}}_{(\lambda_1,\lambda_2)}(P,Q) = \mu^{\mathrm{Q}}_{(0,\lambda_2)}(P,Q).$$

Taken together, these facts prove that the quadratic pseudospectrum is constant.

Since $[P,Q](f) = -if$, Lemma 8.1 says

$$\|Pf\|^2 + \|Qf\|^2 \geq 1.$$

Therefore $\mu^{\mathrm{Q}}_{(\mathbf{0})}(P,Q) \geq 1$ and so

$$\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(P,Q) \geq 1$$

Now consider $g(x) = e^{-\frac{1}{2}x^2}$. Clearly

$$Q^2(g)(x) = x^2 e^{-\frac{1}{2}x^2} = x^2 g(x).$$

We also find

$$\begin{aligned} P^2(g)(x) &= P\left(ixe^{-\frac{1}{2}x^2}\right) \\ &= -i\frac{d}{dx}\left(ixe^{-\frac{1}{2}x^2}\right) \\ &= \frac{d}{dx}\left(xe^{-\frac{1}{2}x^2}\right) \\ &= e^{-\frac{1}{2}x^2} - x^2 e^{-\frac{1}{2}x^2} \\ &= g(x) - x^2 g(x) \end{aligned}$$

and so find

$$\left(P^2 + Q^2\right) g = g.$$

Having found an eigenvector $Q_{(0,0)}(P,Q) = P^2 + Q^2$ we have proven

$$\mu^{\mathrm{Q}}_{\boldsymbol{\lambda}}(P,Q) \leq 1. \quad \square$$

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Appendix A. Calculations for Example 7.3

Here if the rest of the proof for the case $b = 1$. We can solve for x in terms of z on the $e = 0$ curve, so

$$x = \left(z^4 - 3z^3 + 4z^2 - 3z + 1\right)^{1/4}.$$

which simplifies to

$$x = |z-1|^{\frac{1}{2}} \left(z^2 - z + 1\right)^{1/4}.$$

This substitution into $f(x, z)$ where,

$$f(x,z) = x^2 z - x(z-1) + (z-1)\left(z^2 - z + 1\right),$$

leads to the one-variable function

$$f_e(z) = z\,|z-1|\left(z^2-z+1\right)^{1/2} - (z-1)\,|z-1|^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} + (z-1)\left(z^2-z+1\right)$$

For $z \leq 1$ this becomes

$$f_e(z) = z\,(1-z)\left(z^2-z+1\right)^{1/2} + (1-z)\,(1-z)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} - (1-z)\left(z^2-z+1\right).$$

For $0 \leq z \leq 1$ we have $z^2 - z + 1 \leq 1$ so

$$\begin{aligned}
f_e(z) &\geq z\,(1-z)\left(z^2-z+1\right)^{1/2} + (1-z)\,(1-z)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/2} - (1-z)\left(z^2-z+1\right)^{\frac{1}{2}} \\
&= \left(z(1-z) + (1-z)(1-z)^{\frac{1}{2}} - (1-z)\right)\left(z^2-z+1\right)^{\frac{1}{2}} \\
&= (1-z)\left(z + (1-z)^{\frac{1}{2}} - (1-z)\right)\left(z^2-z+1\right)^{\frac{1}{2}} \\
&= (1-z)\left(2z - 1 + (1-z)^{\frac{1}{2}}\right)\left(z^2-z+1\right)^{\frac{1}{2}} \\
&\geq 0.
\end{aligned}$$

For $z \leq 0$ we have $z^2 - z + 1 \geq 1$ so

$$\begin{aligned}
f_e(z) &\leq z\,(1-z)\left(z^2-z+1\right)^{1/4} + (1-z)\,(1-z)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} - (1-z)\left(z^2-z+1\right) \\
&= (1-z)\left(z\left(z^2-z+1\right)^{1/4} + (1-z)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} - \left(z^2-z+1\right)\right) \\
&= (1-z)\left(z^2-z+1\right)^{1/4}\left(z + (1-z)^{\frac{1}{2}} - \left(z^2-z+1\right)^{\frac{3}{4}}\right)
\end{aligned}$$

One can see this is at most zero by looking at its derivative.

For $z \geq 1$ the formula for f_e becomes

$$f_e(z) = z\,(z-1)\left(z^2-z+1\right)^{1/2} - (z-1)\,(z-1)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} + (z-1)\left(z^2-z+1\right).$$

Here $z^2 - z + 1 \geq 1$ so

$$\begin{aligned} f_e(z) &= (z-1)\left(z\left(z^2-z+1\right)^{1/2} - (z-1)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/4} + \left(z^2-z+1\right)\right) \\ &\geq (z-1)\left(z\left(z^2-z+1\right)^{1/2} - (z-1)^{\frac{1}{2}}\left(z^2-z+1\right)^{1/2} + \left(z^2-z+1\right)^{\frac{1}{2}}\right) \\ &= (z-1)\left(z^2-z+1\right)^{1/2}\left(z+1-(z-1)^{\frac{1}{2}}\right) \\ &\geq 0. \end{aligned}$$

Data availability

Scripts to numerically validate Example 7.3 and assist with the calculations in Section 6 are at https://github.com/acerjan/mult_var_pseudospectrum_in_Cstar_algebras. In addition, there the reader can find videos that supplement Figs. 7.1 and 7.2.

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