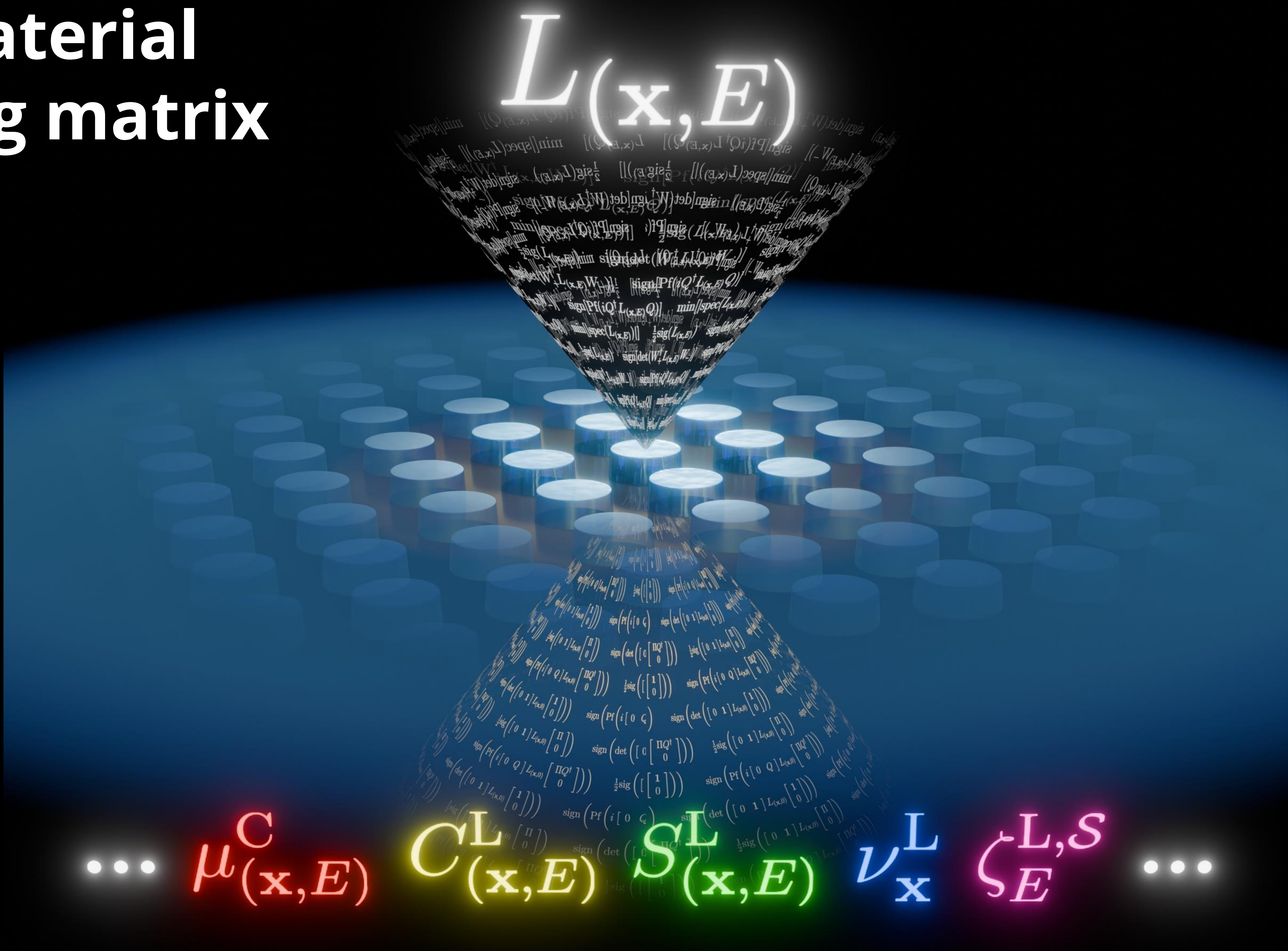
Classifying material topology using matrix homotopy



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Topology and Geometry
Beyond Perfect Crystals

Nordita

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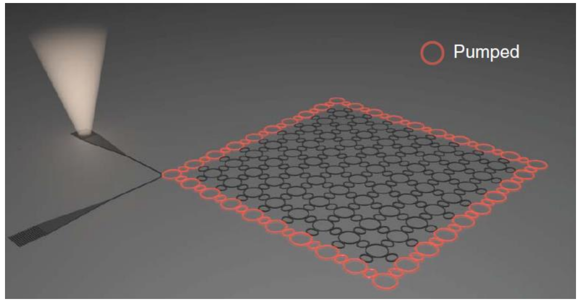
Outline

- An operator-based approach to topological physics
    - Uses a framework called the "***spectral localizer***"
- Emergence of Hofstader's butterfly
- Identifying fragile topology
- Classifying topology in non-linear systems
    - Topological dynamics
- Application directly to Maxwell's equations
    - Incorporating radiative boundaries

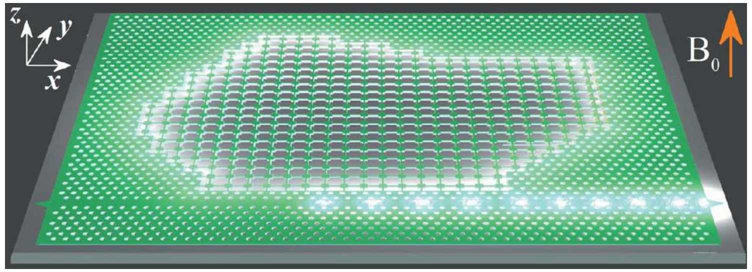
Why make photonics topological?

Topological lasers

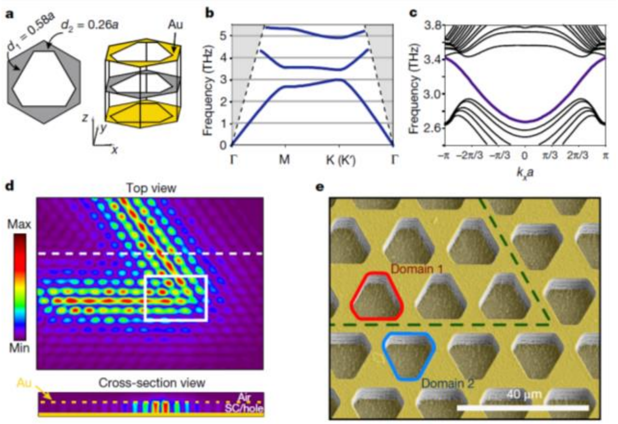
- ➢ Robust against disorder
- ➢ Efficient phase locking



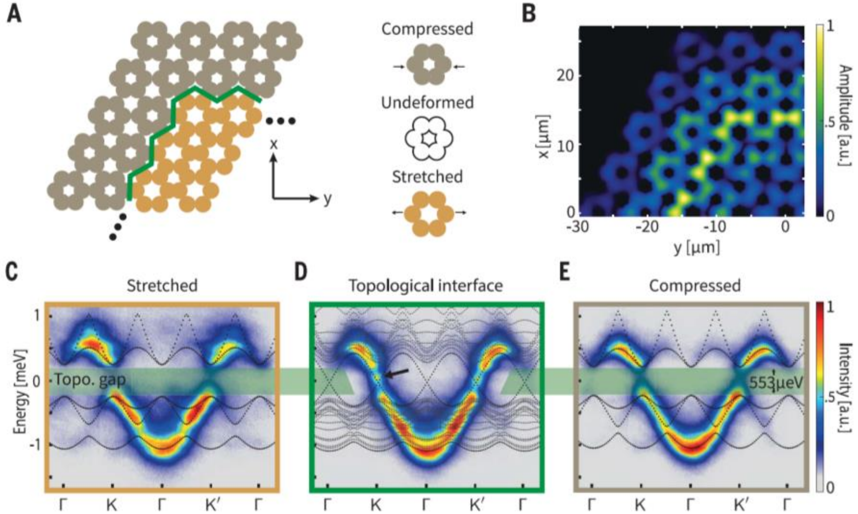
Bandres et al., *Science* **359**, 1231 (2018)
Harari et al., *Science* **359**, eaar4003 (2018)



Bahari et al., *Science* **358**, 636 (2017)



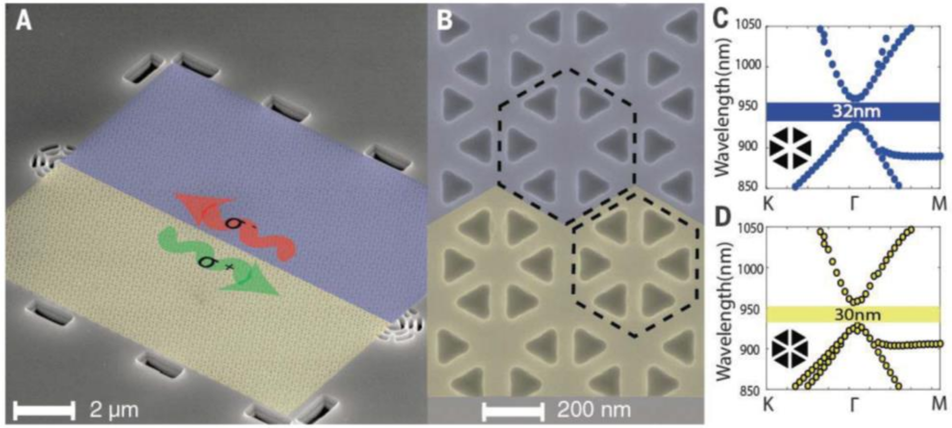
Zeng et al., *Nature* **578**, 246 (2020)



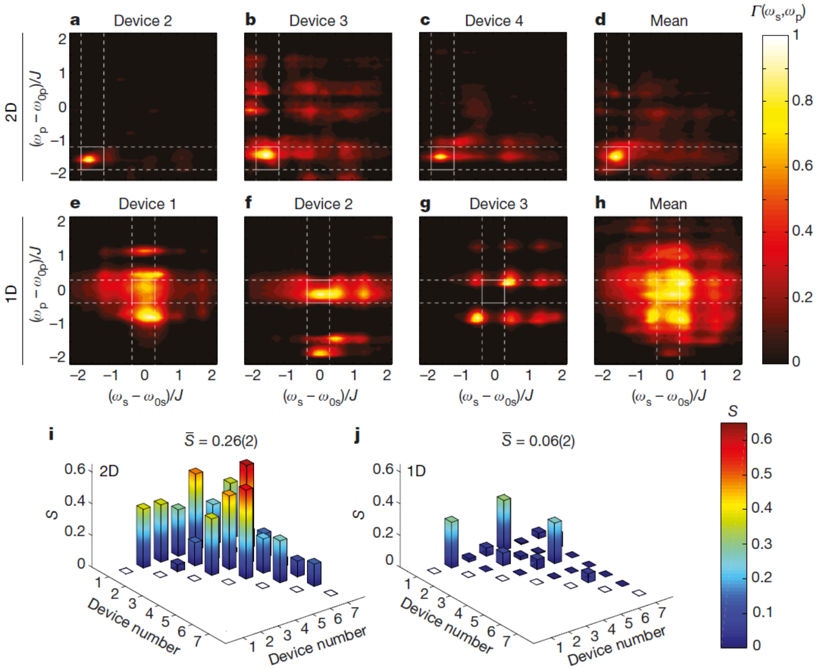
Dikopoltsev et al., *Science* **373**, 1514 (2021)

Why make photonics topological?

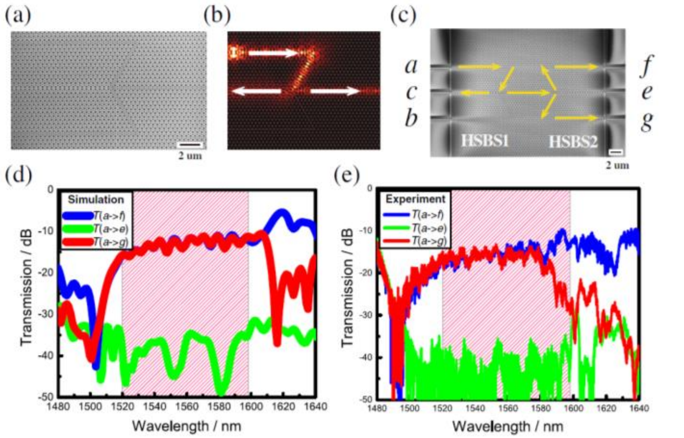
Routing of quantum information



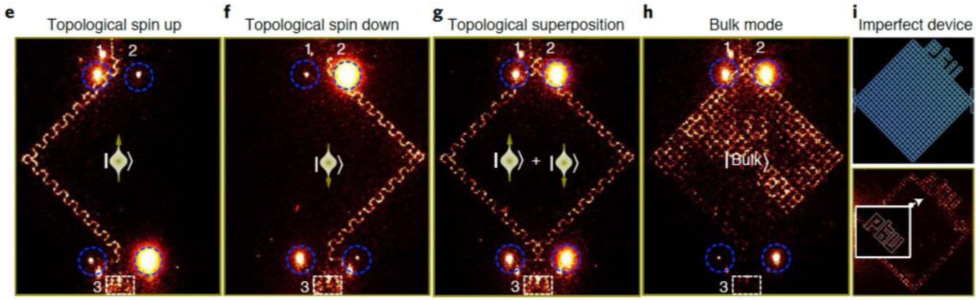
Barik et al., *Science* **359**, 666 (2018)



Mittal et al., *Nature* **561**, 502 (2018)



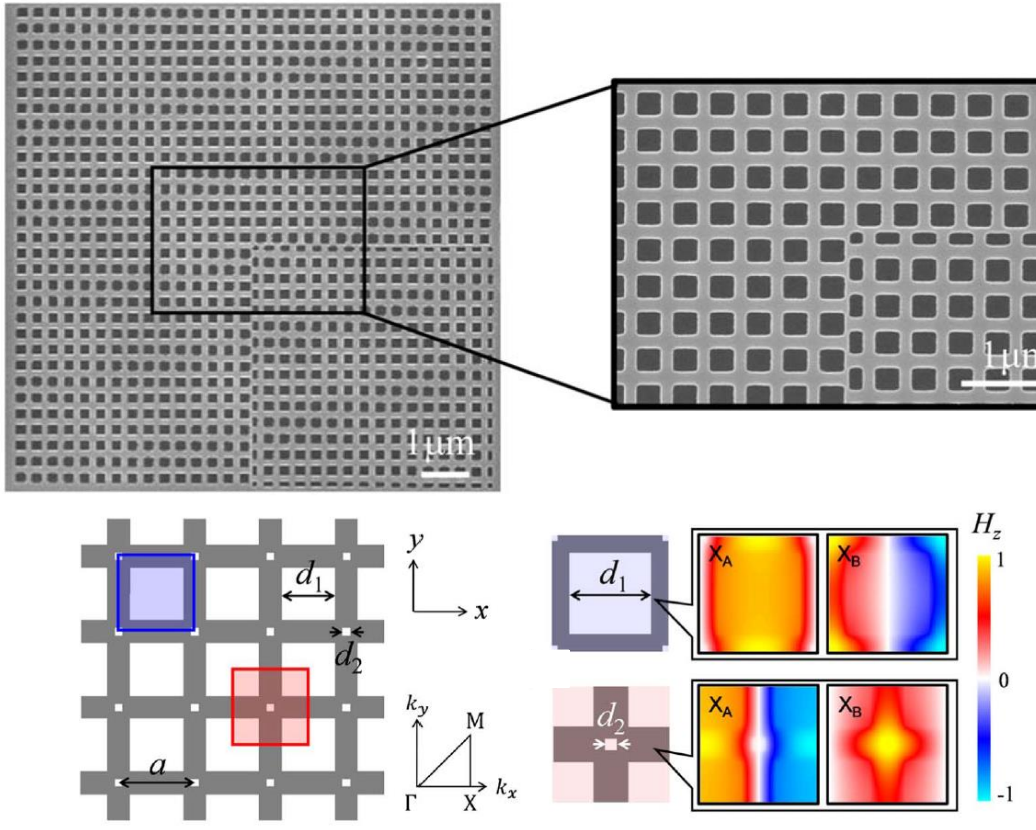
Chen et al., *Phys. Rev. Lett.* **126**, 230503 (2021)



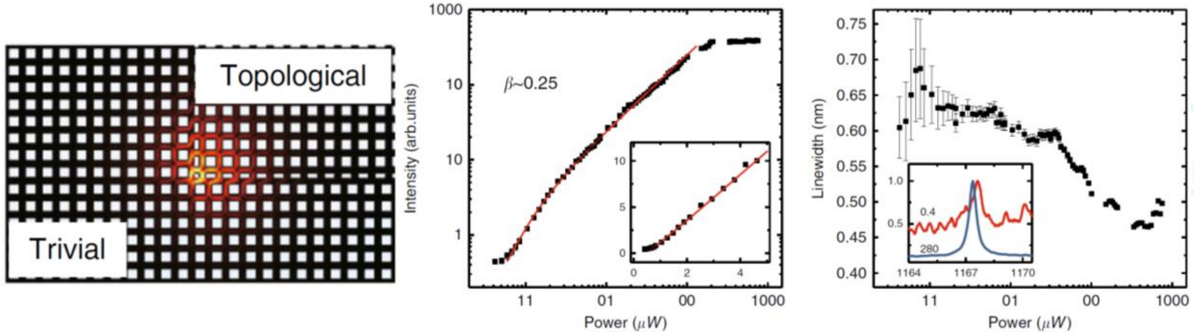
Dai et al., *Nat. Photonics* **16**, 248 (2022)

Why make photonics topological?

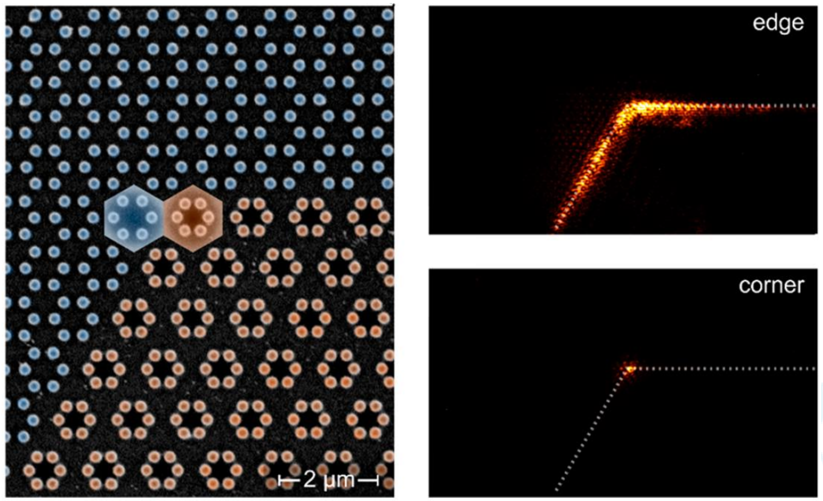
Creating cavities for light-matter interaction



Ota et al., *Optica* **6**, 786 (2019)



Zhang et al., *Light Sci. Appl.* **9**, 109 (2020)



Smirnova et al., *Phys. Rev. Lett.*, **123**, 103901 (2019)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Challenges with invariants

Where band theory can be applied, band theory is awesome.

Where is band theory not applicable (or not useful)?

1) **Material lacks translational symmetry**
    - Quasicrystals
    - Amorphous materials
    - Disorder
    - Finite size effects

2) **Heterostructure lacks a complete or incomplete band gap**
    - ➢ Band theory is applicable, but…
        - ➢ Not always clear how to calculate the invariant
        - ➢ No measure of protection

3) **System is non-linear**
    - Localized response breaks translational symmetry

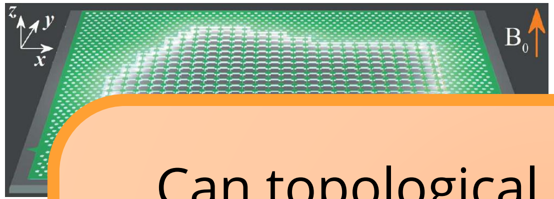
Challenges with invariants in photonics

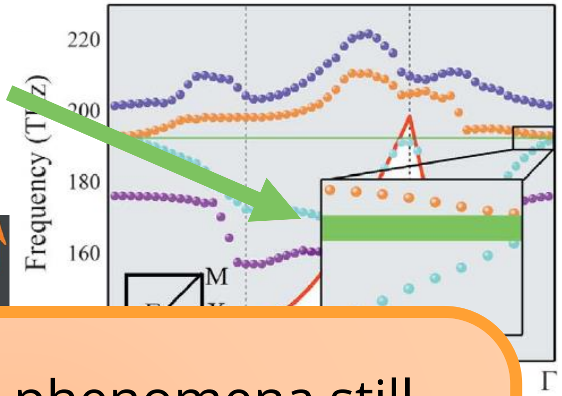
We'd like nanophotonic Chern insulators

- ➢ Non-reciprocal edge states

But… it's hard to break time-reversal symmetry

Vanishing bandgap (42 pm)

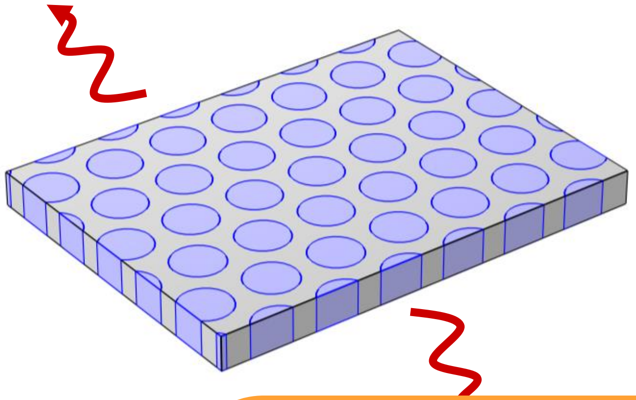


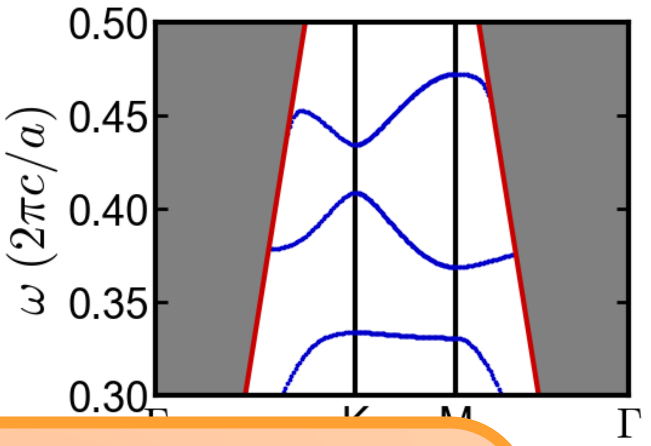


Can topological phenomena still manifest without a complete band gap?

- ➢ Chiral edge resonance?

Related challenge: photonic crystal slabs and metasurfaces radiate out-of-plane





Can resonances and bound states be mixed in formula for topological invariants?

$$C_n = \frac{1}{2\pi}\int_{BZ} \nabla_{\mathbf{k}} \times i\,\langle\psi_{n\mathbf{k}}|\nabla_{\mathbf{k}}|\psi_{n\mathbf{k}}\rangle d^2\mathbf{k}$$

Challenges with invariants in photonics

No current theory for finite systems

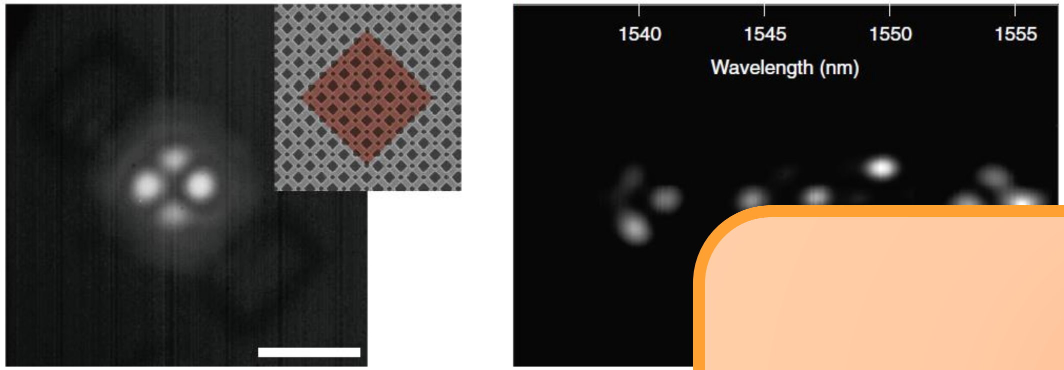
How close can two topological cavities be, while maintaining protection?

Or how close can two chiral edge states be in a topological Chern system?

Estimate:

$$e^{-\frac{x}{L}}$$

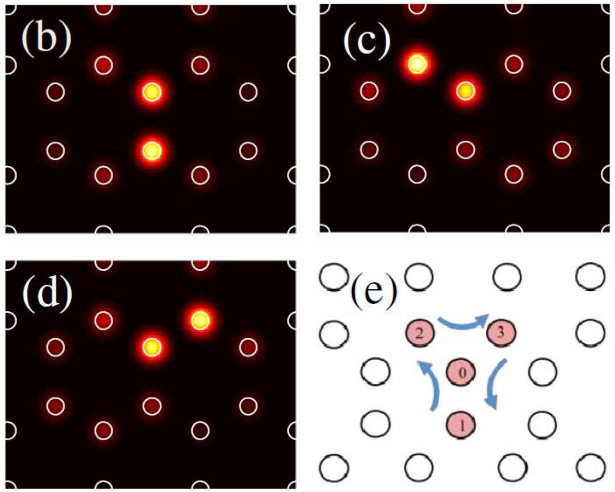
Decay length L set by band gap width ΔE



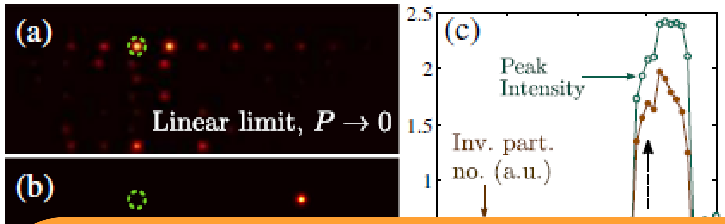
Kim et al., *Nat. Commun.* **11**,

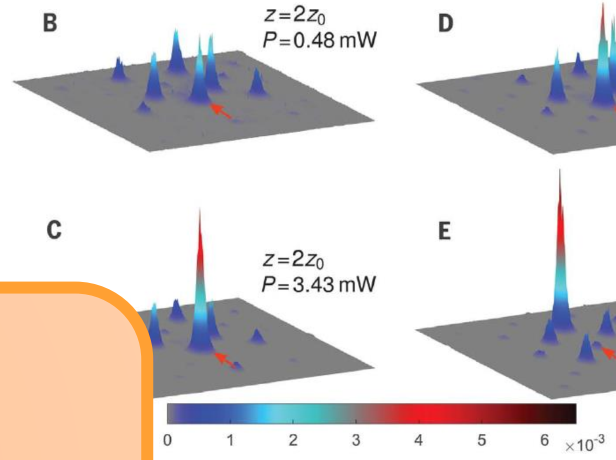
Is there a local measure of topological protection?

Photonic non-linearities are local



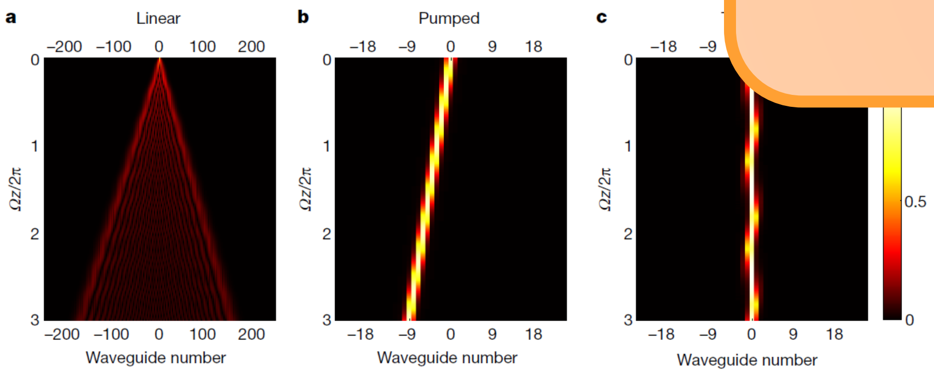
Lumer et al., *Phys. Rev. Lett.* **111**, 243905 (2013)



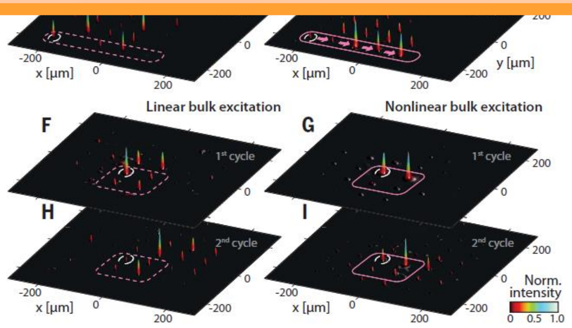


and Rechtsman, *Science* **368**, 856 (2020)

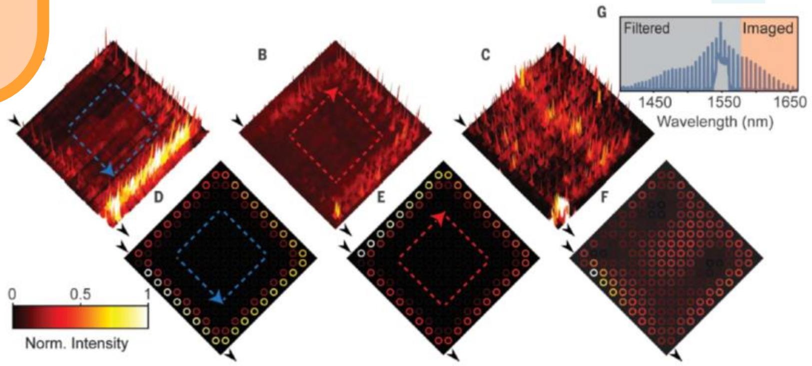
Can a topological invariant be defined without a bulk?



Jürgensen et al., *Nature* **596**, 63 (2021)
Jürgensen et al., *Nat. Phys.* **19**, 420 (2023)

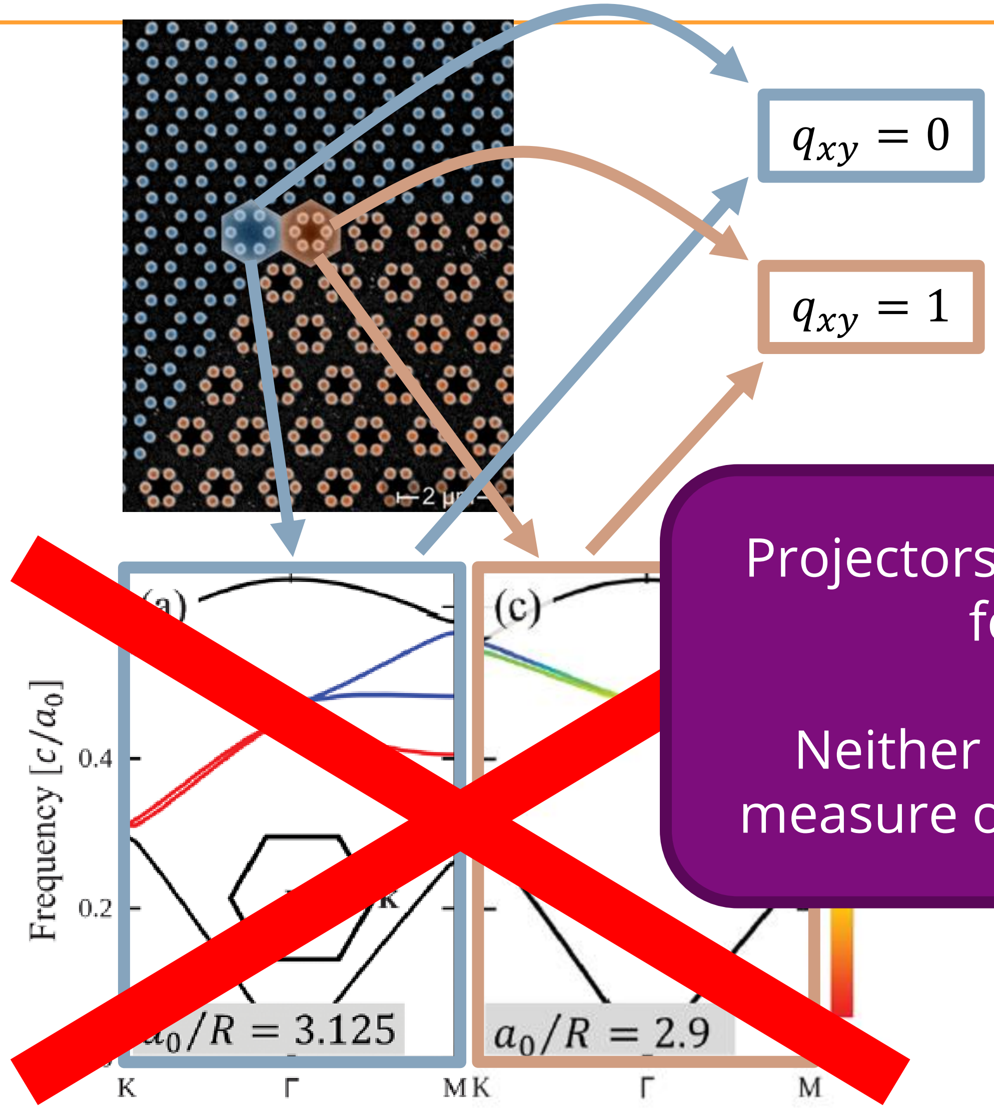


Maczewsky et al., *Science* **370**, 701 (2020)



Flower et al., *Science* **384**, 1356 (2024)

Local real-space approaches to material topology



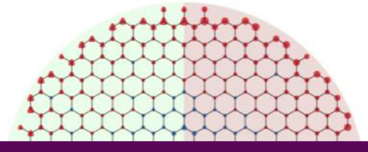
$q_{xy} = 0$

$q_{xy} = 1$

Wu and Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Kitaev:

$$\nu(P) = 12\pi i \sum_{j\in A}\sum_{k\in B}\sum_{l\in C}\left(P_{jk}P_{kl}P_{lj} - P_{jl}P_{lk}P_{kj}\right)$$



Kitaev, *Ann. Phys.* **321**, 2 (2006)
Mitchell et al., *Nat. Phys.* **14**, 380 (2018)

Projectors are difficult to calculate for real systems

Neither framework has a local measure of topological protection.

$$(\mathbf{r}',\mathbf{r}) - \tilde{Y}(\mathbf{r},\mathbf{r}')\tilde{X}(\mathbf{r}',\mathbf{r})]\,d\mathbf{r}'$$

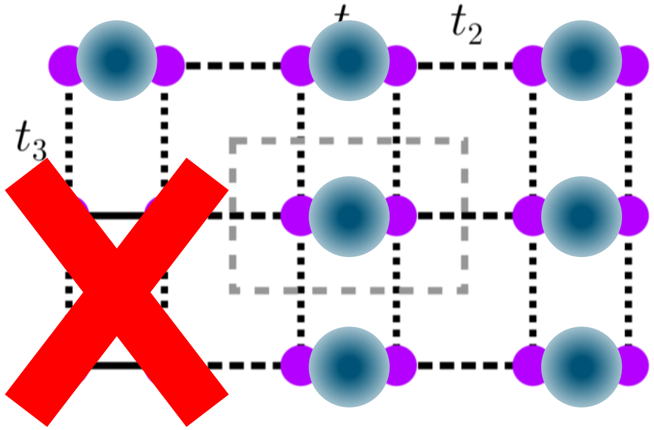
$$\tilde{X}(\mathbf{r},\mathbf{r}') = \int P(\mathbf{r},\mathbf{r}'')x''P(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''$$

Bianco and Resta, *Phys. Rev. B* **84**, 241106(R) (2011)

Implications of topology on the Wannier basis

Systems with non-trivial Chern numbers **DO NOT** possess a complete localized Wannier basis.

$$C_n = \frac{1}{2\pi}\int_{BZ}\left(\frac{\partial A_y^n}{\partial k_x} - \frac{\partial A_x^n}{\partial k_y}\right) d^2\mathbf{k} \neq 0 \qquad \Longleftrightarrow$$



This is an ***if and only if*** statement

- ➢ No complete localized Wannier basis necessitates a non-trivial Chern number.

This generalizes to many other classes of topology

Example: No localized Wannier basis that respects time-reversal symmetry
⟺ non-trivial Kane-Mele invariant (Quantum spin Hall)

Brouder et al., *Phys. Rev. Lett.* **98**, 046402 (2007)

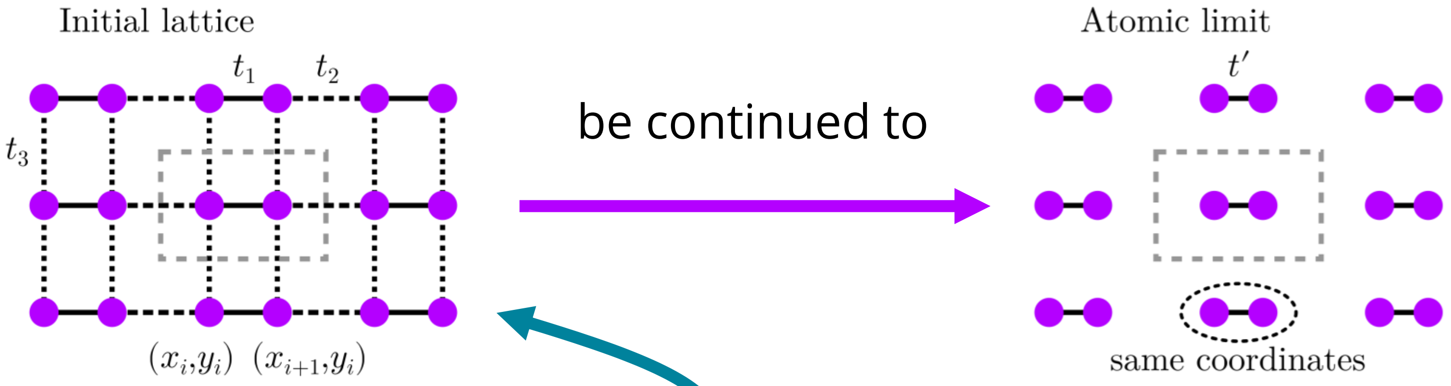
Soluyanov and Vanderbilt, *Phys. Rev. B* **83**, 035108 (2011)

Topology as "Wannierizability"

Instead of an invariant, "Does the system possess a complete Wannier basis?"

- Band gap stays open
- Symmetries are preserved

Can a lattice



be continued to

without violating?

If yes
➢ Trivial

If no
➢ Topological

In other words, "Can the system be permuted to an *atomic limit*?"

(and if multiple inequivalent limits exist, which one?)

➢ Can answer using a lattice's band structure

➢ **Topological quantum chemistry**

Bradlyn et al., *Nature* **547**, 298 (2017)

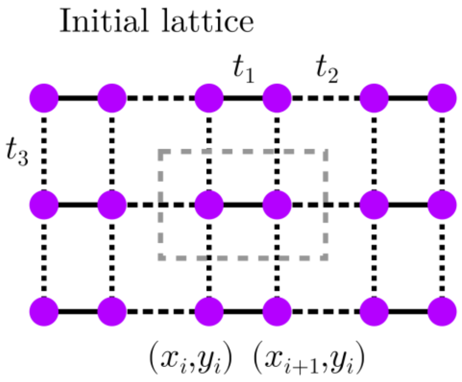
Kitaev, *AIP Conference Proceedings* **1134**, 22 (2009)
Hastings and Loring, *Ann. Phys.* **326** 1699 (2011)
Taherinejad et al., *Phys. Rev. B* **89**, 115102 (2014)
Kruthoff et al., *Phys. Rev. X* **7**, 041069 (2017)
Po et al., *Nat. Commun.* **8**, 50 (2017)

Topology as an atomic limit

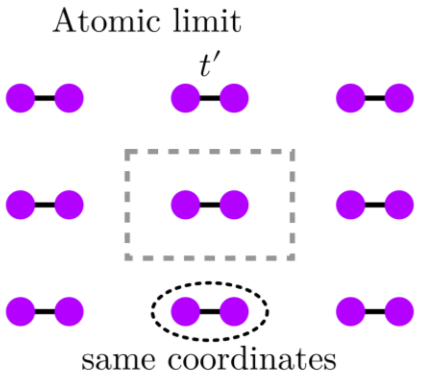
Instead of an invariant, “Can the system be permuted to an *atomic limit*?”

- Band gap stays open
- Symmetries are preserved

Can a lattice



be continued to



without violating?

If yes
  - ➢ Trivial

If no
  - ➢ Topological

Can the operators

$$H = \begin{bmatrix} \ddots & -t_2 & & -t_3 & & \\ -t_2 & \varepsilon & -t_1 & & -t_3 & \\ & -t_1 & \varepsilon & -t_2 & & -t_3 \\ -t_3 & & -t_2 & \varepsilon & -t_1 & \\ & -t_3 & & -t_1 & \varepsilon & -t_2 \\ & & -t_3 & & -t_2 & \ddots \end{bmatrix}$$

$$\boxed{[H, X] \neq 0}$$

$$X = \begin{bmatrix} \ddots & & & & & \\ & x_{i-1} & & & & \\ & & x_i & & & \\ & & & x_{i+1} & & \\ & & & & x_{i+2} & \\ & & & & & \ddots \end{bmatrix}$$

be continued to

$$\boxed{\left[H^{(\mathrm{AL})}, X^{(\mathrm{AL})}\right] = 0}$$

$$H_a = \begin{bmatrix} \ddots & & & & & \\ & \varepsilon' & -t' & & & \\ & -t' & \varepsilon' & & & \\ & & & \varepsilon' & -t' & \\ & & & -t' & \varepsilon' & \\ & & & & & \ddots \end{bmatrix}$$

$$X_a = \begin{bmatrix} \ddots & & & & & \\ & x_i' & & & & \\ & & x_i' & & & \\ & & & x_{i+1}' & & \\ & & & & x_{i+1}' & \\ & & & & & \ddots \end{bmatrix}$$

without violating similar restrictions?

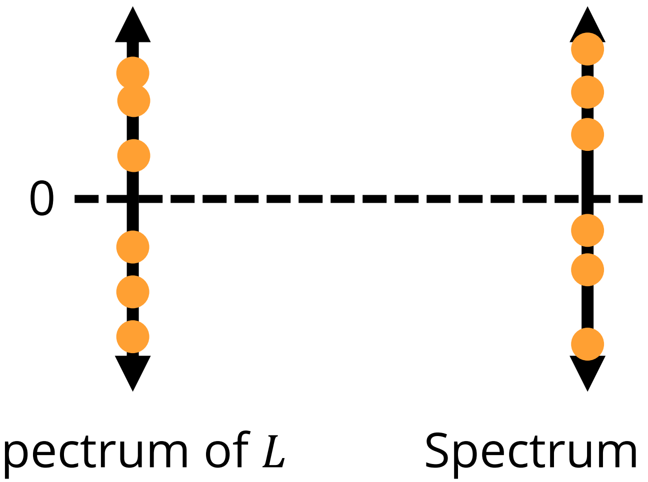
Topology from operators

Instead of an invariant, “Can the system be permuted to an *atomic limit*?”

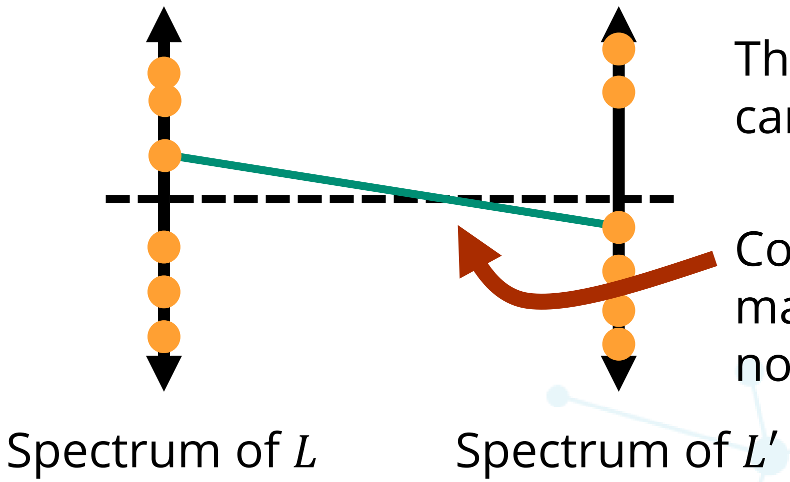
“Can the system’s operators be permuted to be *commuting*?”

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices <u>if and only if</u> $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.



These matrices can be so connected



These matrices cannot

Connecting matrix becomes non-invertible

Topology from operators

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices if and only if $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.

Theorem (Choi, 1988): If R and S are n-by-n matrices with $RS = SR$, then

$$\mathrm{sig}\begin{bmatrix} R & S \\ S^{\dagger} & -R \end{bmatrix} = 0$$

How do these results help?

- $R \to (H - EI)$
- $S \to \kappa(X - xI) - i\kappa(Y - yI)$

And the requirement that $RS = SR$ becomes

$$[H - EI, X - xI] = 0 \text{ and } [H - EI, Y - yI] = 0$$

Construct the 2D *spectral localizer*:

$$L_{(x,y,E)}(X, Y, H) = \begin{bmatrix} H - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - EI) \end{bmatrix}$$

If $\mathrm{sig}(L_{(x,y,E)}(X, Y, H)) = 0$ for a given E, x, y, then

the system can be continued to the atomic limit at that point.

Topology from operators

Intuitively… what's going on?

$$L_{(x,y,E)}(X,Y,H)$$
$$= \kappa(X - xI) \otimes \sigma_x + \kappa(Y - yI) \otimes \sigma_y + (H - EI) \otimes \sigma_z$$

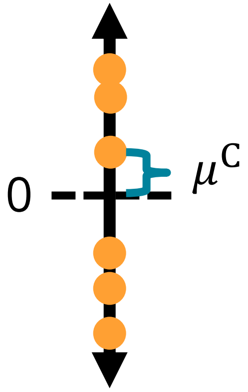
- H and X, Y contain "orthogonal" information
- Pauli matrices (+ identity) form a basis for 2-by-2 Hermitian matrices.

➢ Combination preserves the independent information in H and X, Y while forming a single matrix.

Measure of protection (i.e., a "local gap")

$$\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)} = \sigma_{\min}[L_{(x_1,\ldots,x_d,E)}(X_1,\ldots,X_d,H)]$$

(smallest eigenvalue of $L_{(x_1,\ldots,x_d,E)}$)



Spectrum of L

Rigorously,

$$\|\delta H\| < \mu^C$$

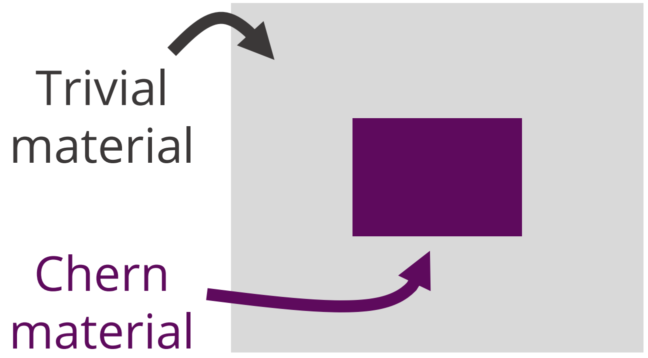
cannot change local topology

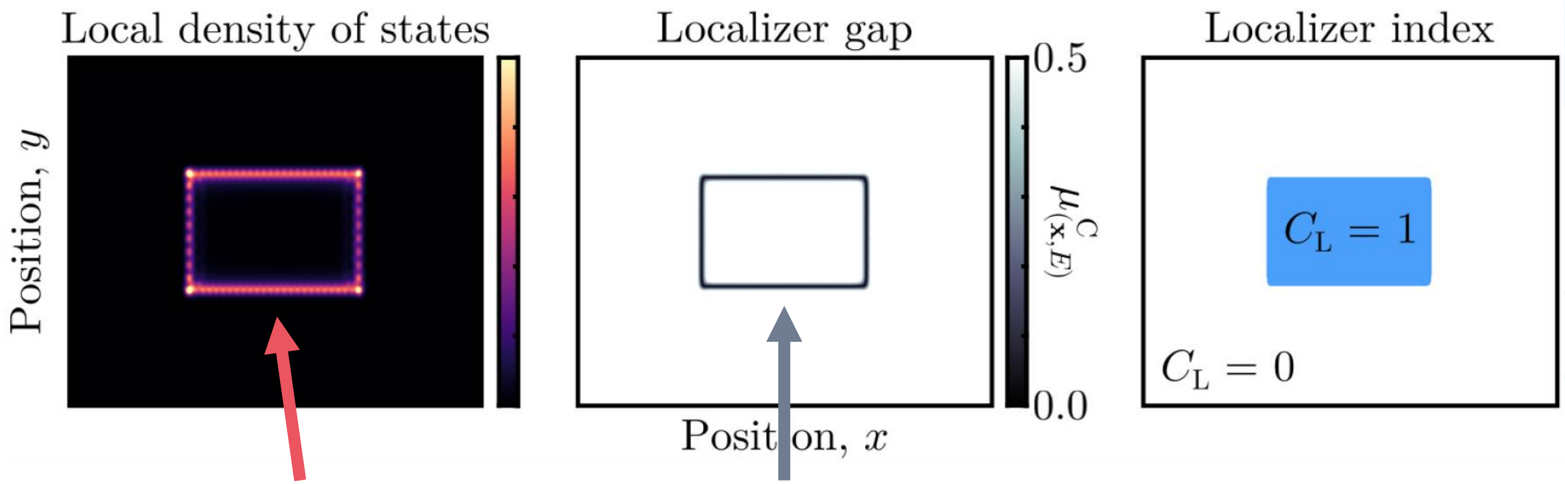
(Weyl's inequality)

Loring, *Ann. Phys.* **356**, 383 (2015)
Loring and Schulz-Baldes, *New York J. Math.* **23**, 1111 (2017)
Loring and Schulz-Baldes, *J. Noncommut. Geom.* **14**, 1 (2020)

What does this look like?

Topological heterostructure



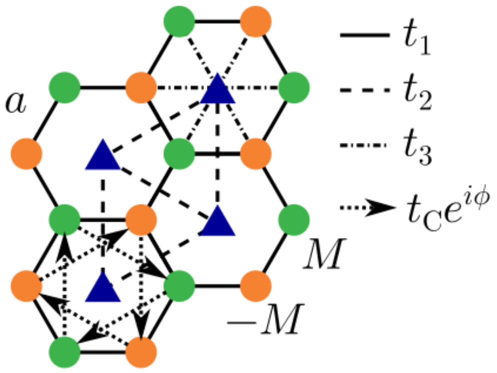


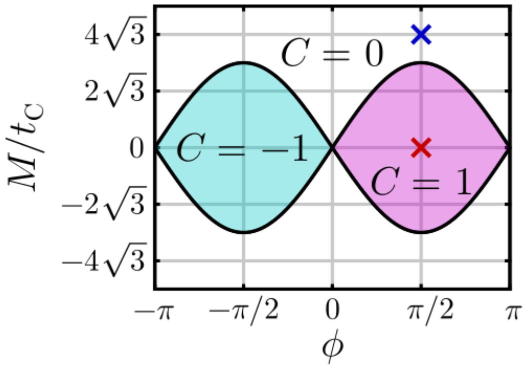
Connection between chiral edge states and local gap closing?

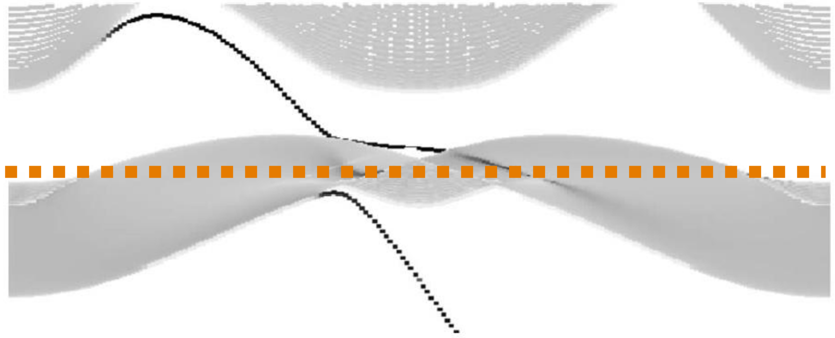
- **YES!!!**
  - Built-in bulk-boundary correspondence
  - Gap closings *necessitate* nearby states of the Hamiltonian

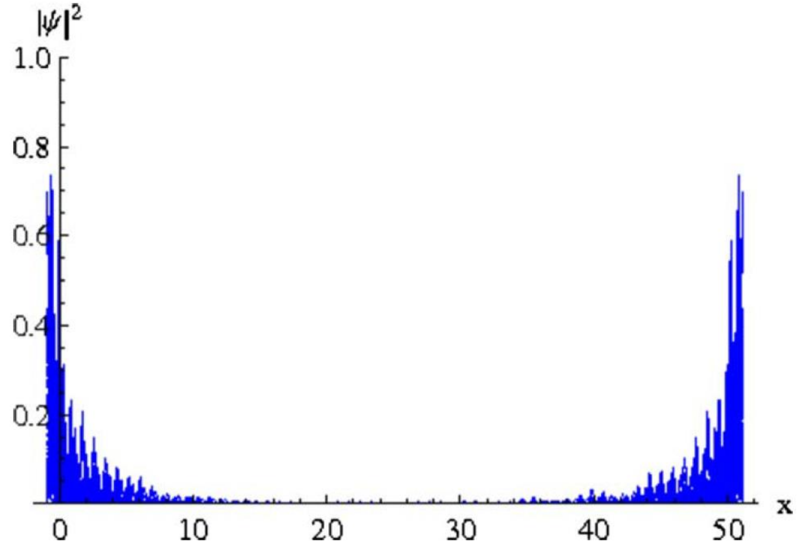
Topology for a gapless system?

“metallized Haldane model”









Found boundary-localized states

- ➢ Resistant to hybridization
- ➢ Robust against mild disorder

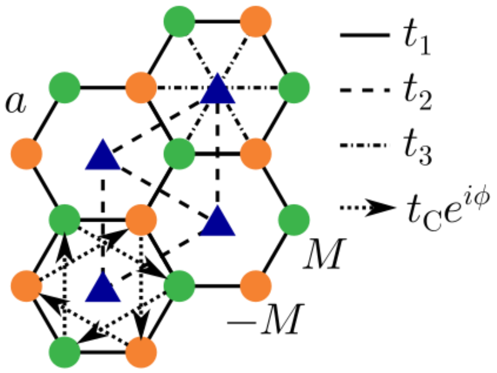
Can such systems be classified?

- What if Fermi energy is **here**?
- Measure of topological protection without a band gap?

D. Hsieh et al., *Science* **323**, 919 (2009)
Bergman and Refael, *Phys. Rev. B* **82**, 195417 (2010)
Junck et al., *Phys. Rev. B* **87**, 235114 (2013)

Chern metal

“metallized Haldane model”

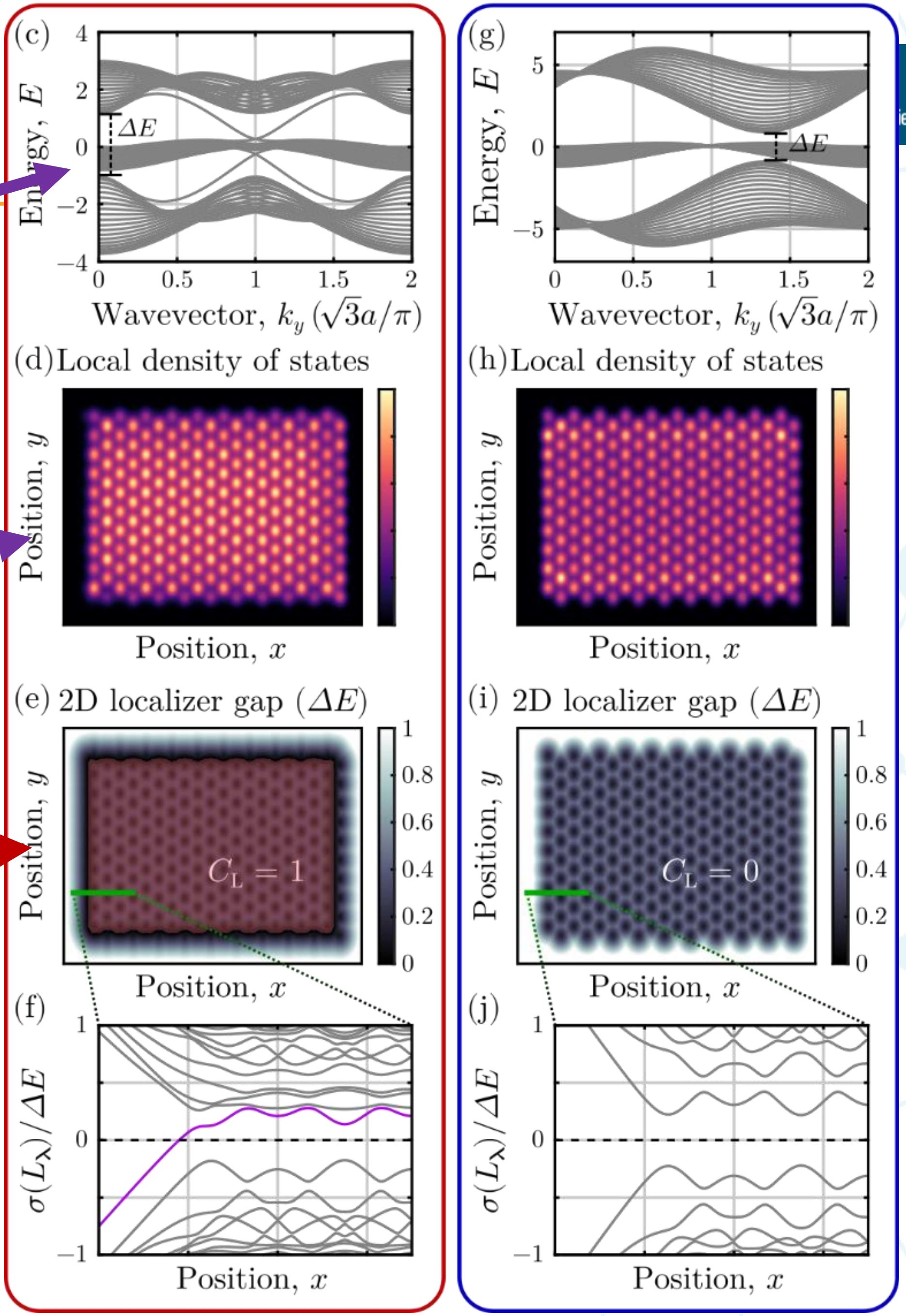


Ribbon band structure

No qualitative difference in LDOS at $E = 0$

Even though $H - E_F I$ has eigenvalues at 0

$L_{(x,y,E)}$ can still be gapped!



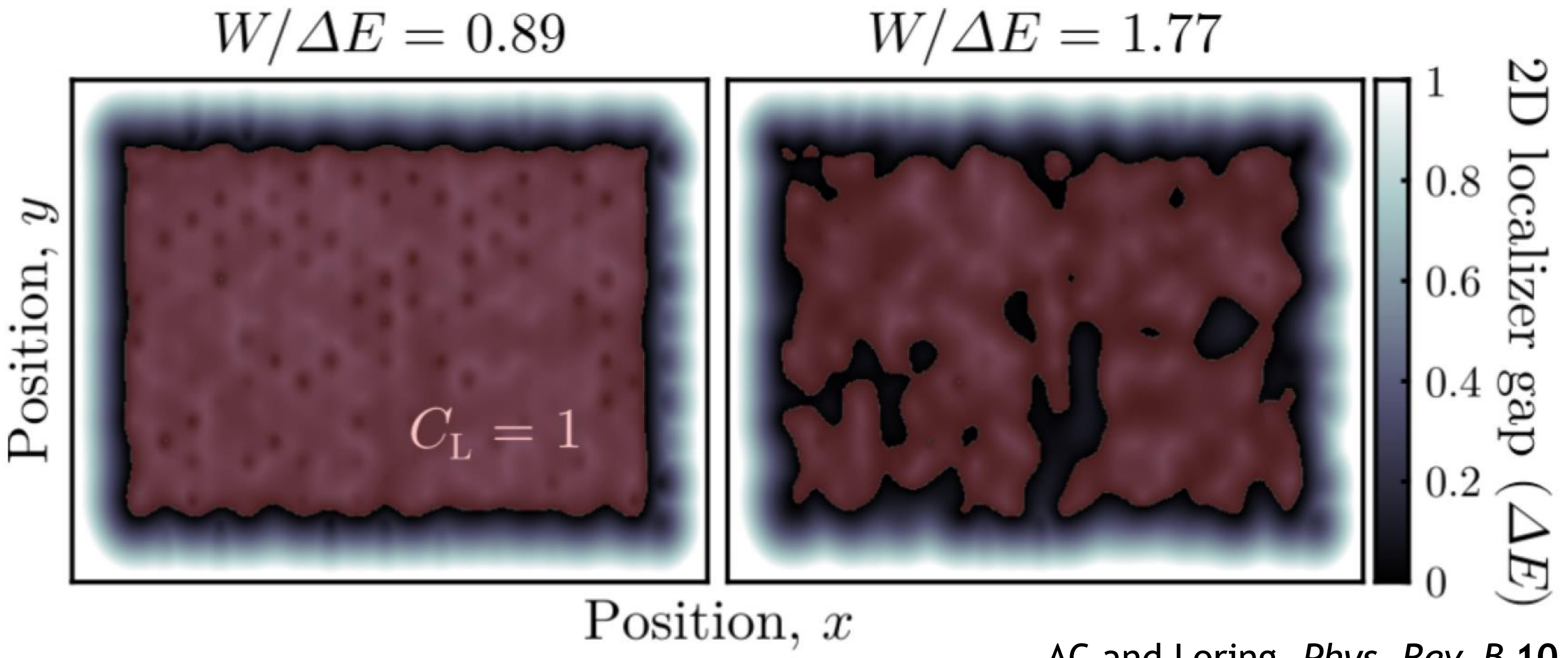
Bergman and Refael, *Phys. Rev. B* **82**, 195417 (2010)
Junck et al., *Phys. Rev. B* **87**, 235114 (2013)

AC and Loring, *Phys. Rev. B* **106**, 064109 (2022)

Disordered Chern metal

By retaining position information from X, Y:

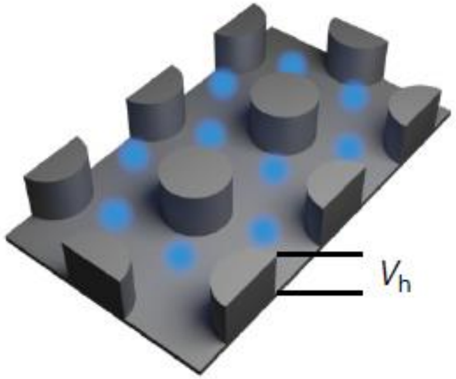
- ➢ Identify local gaps
- ➢ Classify local topology

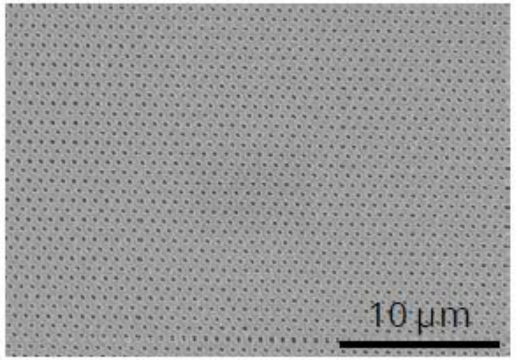


AC and Loring, *Phys. Rev. B* **106**, 064109 (2022)

Application to 2D electron gasses and artificial graphene

Artificial Graphene –
quantum well with added potential V_h



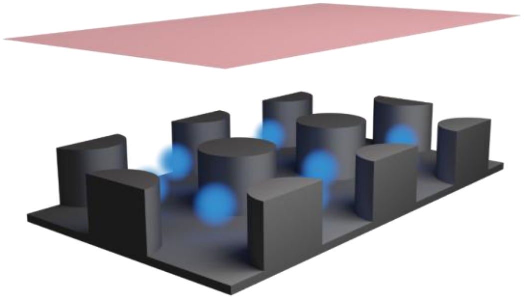


AlSb/InAs/AlSb

$$H = \frac{1}{2m^*}\left(-i\hbar\nabla + e\mathbf{A}(\mathbf{x})\right)^2 + V(\mathbf{x}) - \frac{\mu_B g}{\hbar} s_z B$$

$E_F \approx 4V_h$

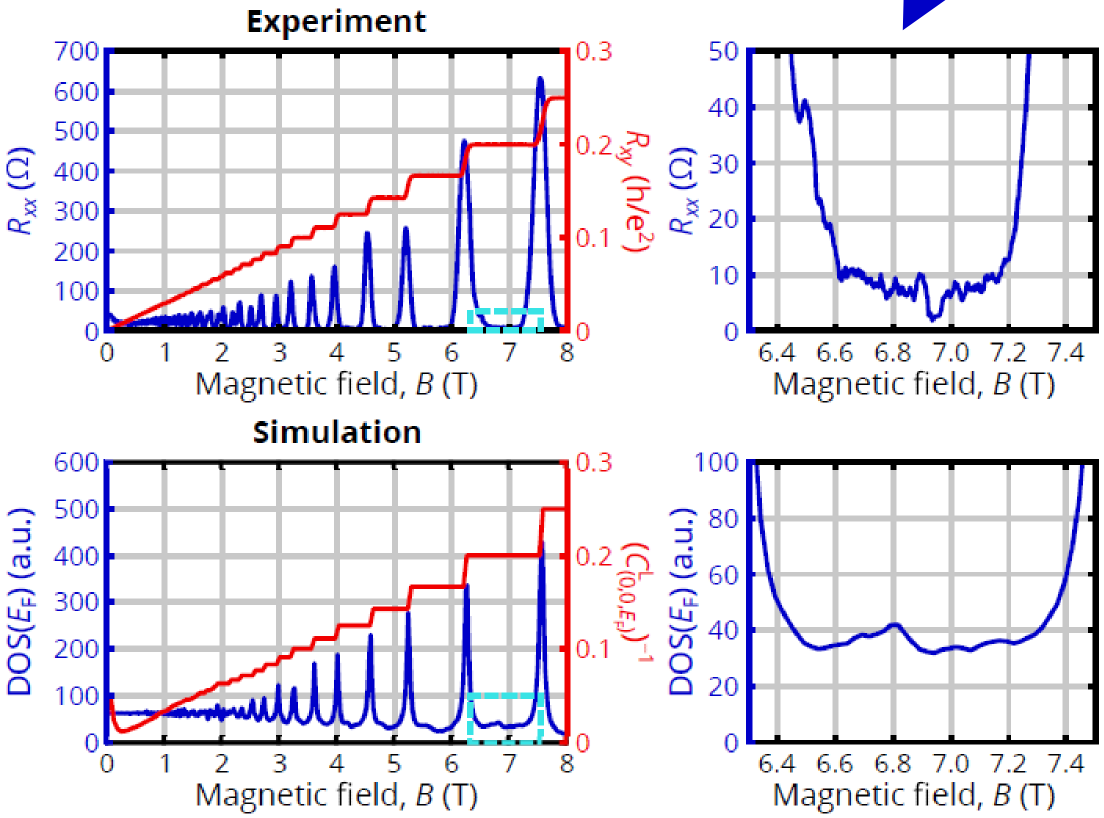
- ➢ System mostly behaves as 2D electron gas
  - ➢ IQHE



Park et al., *Nano Lett.* **8**, 2920 (2008)
Wunsch et al., *New J. Phys.* **10**, 103027 (2008)

Added potential closes the Landau level gaps

Nevertheless, spectral localizer yields correct Hall resistivity



Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Topological origins of pinned states

$$L_{(x,y,E)}(X,Y,H)$$
$$= \kappa(X - xI) \otimes \sigma_x + \kappa(Y - yI) \otimes \sigma_y + (H - EI) \otimes \sigma_z$$

To probe system-level phenomena characterized by length L

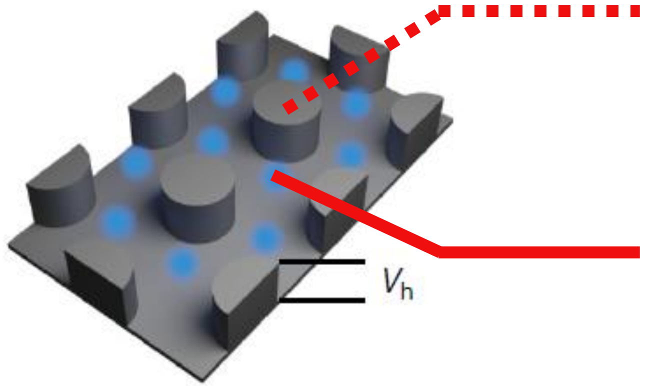
$$\kappa \sim \frac{E_{\text{gap}}}{L}$$

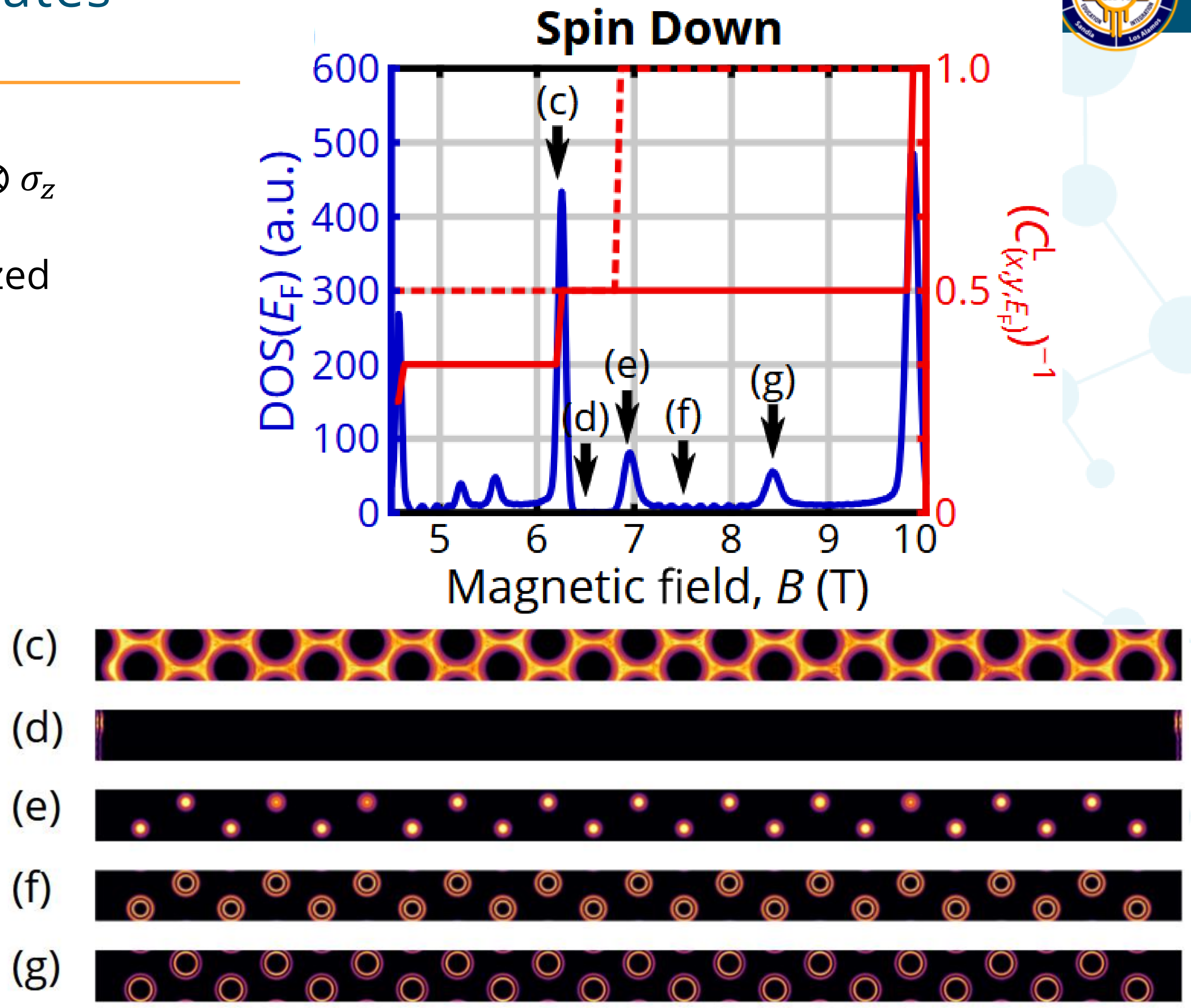
To probe antidot phenomena with diameter D

$$\kappa \sim \frac{E_{\text{gap}}}{D}$$

trading spectral resolution for spatial resolution

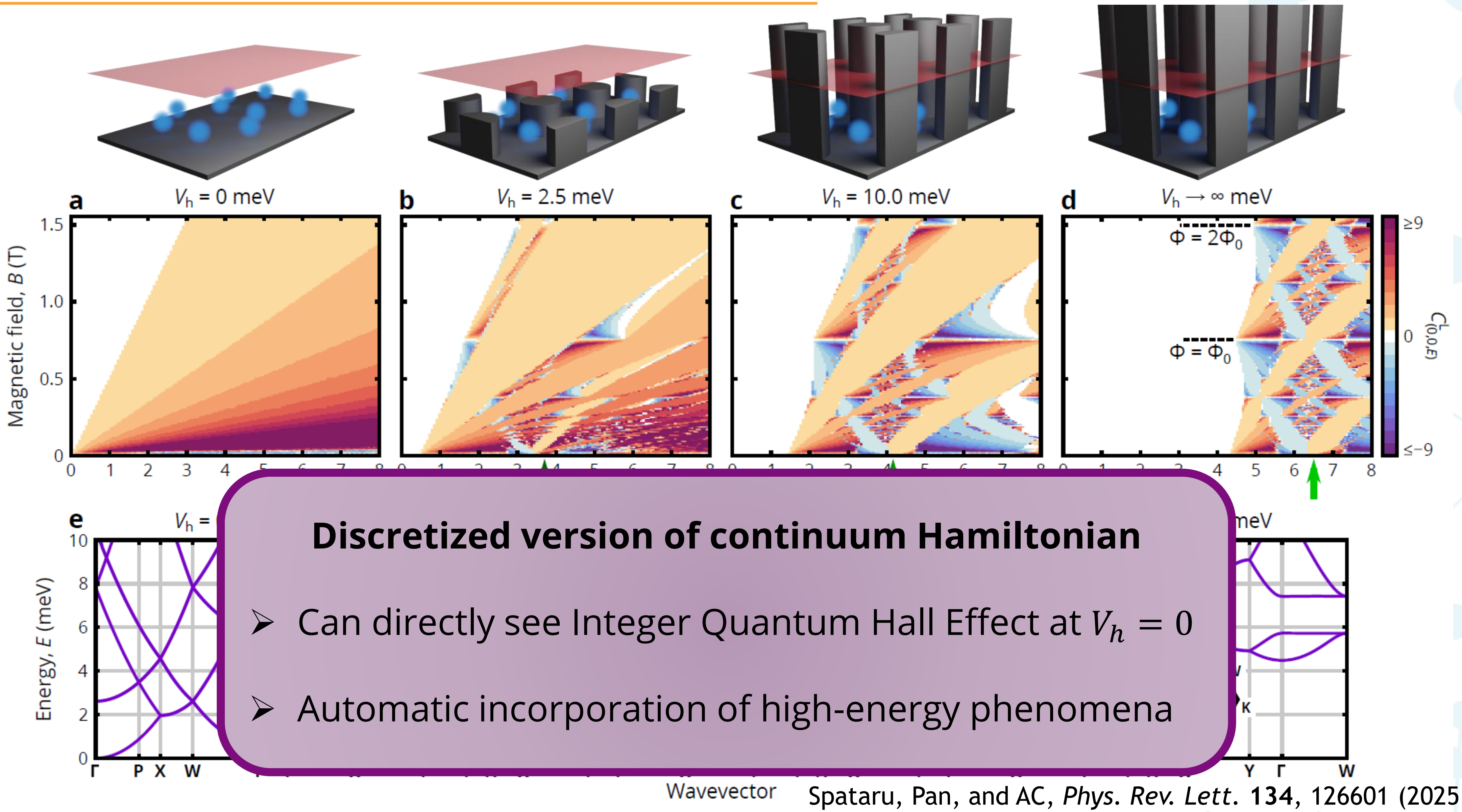
→ requires a larger E_{gap}





Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Emergence of Hofstader’s butterfly as potential is turned on



Discretized version of continuum Hamiltonian

- Can directly see Integer Quantum Hall Effect at $V_h = 0$
- Automatic incorporation of high-energy phenomena

Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Classifying fragile topology via matrix homotopy

Consider a finite 2D system with open boundaries

Hamiltonian H
Position operators X, Y

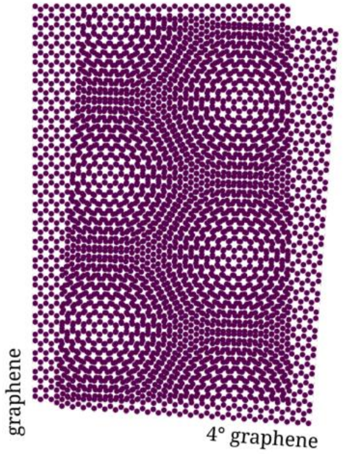
$$H, X, Y \in \mathbf{M}_{2n}(\mathbb{C})$$

Fragile topology can be protected by $(C_2\mathcal{T})$-symmetry

C_2 – 180° rotation about out-of-plane axis
$\mathcal{T}$ – Bosonic time-reversal symmetry, $\mathcal{T}^2 = I$

For a system with this symmetry

$$(C_2\mathcal{T})^{-1}H(C_2\mathcal{T}) = H$$
$$(C_2\mathcal{T})^{-1}X(C_2\mathcal{T}) = -X$$
$$(C_2\mathcal{T})^{-1}Y(C_2\mathcal{T}) = -Y$$



Ponor, Wikimedia Commons

Define

$$M^{\rho} = (C_2\mathcal{T})^{-1}M^{\dagger}(C_2\mathcal{T})$$

after simplifying

$$M^{\rho} = C_2 M^{\top} C_2$$

ρ defines a real structure for the C*-algebra formed by $\mathbf{M}_{2n}(\mathbb{C})$

$$H^{\rho} = H$$
$$X^{\rho} = -X$$
$$Y^{\rho} = -Y$$

Lee, Wong, Vaidya, Loring, and AC, arXiv:2503.03948.

Classifying fragile topology via matrix homotopy

Define

$$M^{\rho} = (C_2\mathcal{T})^{-1}M^{\dagger}(C_2\mathcal{T})$$

after simplifying

$$M^{\rho} = C_2 M^{\top} C_2$$

ρ defines a real structure for the C*-algebra formed by $\mathbf{M}_{2n}(\mathbb{C})$

$$H^{\rho} = H$$
$$X^{\rho} = -X$$
$$Y^{\rho} = -Y$$

In some basis, $\rho \to \mathrm{T}$

Can directly verify that the unitary

$$W = \frac{1}{\sqrt{2}}(C_2 + iI)$$

yields

$$WM^{\rho}W^{\dagger} = (WMW^{\dagger})^{\top}$$

And thus

$$(WHW^{\dagger})^{\top} = WHW^{\dagger} \longleftarrow \text{symmetric}$$

$$\left.\begin{aligned}(WXW^{\dagger})^{\top} &= -WXW^{\dagger}\\ (WYW^{\dagger})^{\top} &= -WYW^{\dagger}\end{aligned}\right\} \text{skew symmetric}$$

Lee, Wong, Vaidya, Loring, and AC, arXiv:2503.03948.

Homotopy invariant of skew symmetric matrices

$$T = \begin{bmatrix} 0 & \alpha_1 & & & & & & \\ -\alpha_1 & 0 & & & & & & \\ & & 0 & \alpha_2 & & & & \\ & & -\alpha_2 & 0 & & & & \\ & & & & & \ddots & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \alpha_n \\ & & & & & & -\alpha_n & 0 \end{bmatrix}$$

If we want to change $\operatorname{sign}[\mathrm{Pf}[T]]$

while preserving $T^\top = -T$

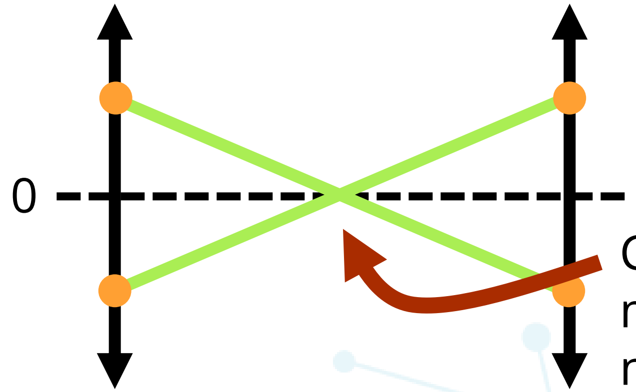
$$\begin{bmatrix} \ddots & & & \\ & 0 & \alpha_j & \\ & -\alpha_j & 0 & \\ & & & \ddots \end{bmatrix} \to \begin{bmatrix} \ddots & & & \\ & 0 & -\alpha_j & \\ & \alpha_j & 0 & \\ & & & \ddots \end{bmatrix}$$

Skew symmetric — $T^\top = -T$

Well-defined sign

Pfaffian — $\mathrm{Pf}[T] = \alpha_1 \alpha_2 \cdots \alpha_n$

Determinant — $\det[T] = \mathrm{Pf}[T]^2$



Connecting matrix becomes non-invertible

Classifying fragile topology via matrix homotopy

$$\begin{aligned} (WHW^\dagger)^\top &= WHW^\dagger & \sigma_y^\top &= -\sigma_y \\ (WXW^\dagger)^\top &= -WXW^\dagger & \sigma_x^\top &= \sigma_x \\ (WYW^\dagger)^\top &= -WYW^\dagger & \sigma_z^\top &= \sigma_z \end{aligned}$$

Form a (nearly) skew-symmetric spectral localizer

$$L_{(x,y,E)}(WXW^\dagger, WYW^\dagger, WHW^\dagger) = \kappa(WXW^\dagger - x) \otimes \sigma_x + \kappa(WYW^\dagger - y) \otimes \sigma_z + (WHW^\dagger - E) \otimes \sigma_y$$

$$= \begin{bmatrix} \kappa(WYW^\dagger - y) & \kappa(WXW^\dagger - x) - i(WHW^\dagger - E) \\ \kappa(WXW^\dagger - x) + i(WHW^\dagger - E) & -\kappa(WYW^\dagger - y) \end{bmatrix}$$

At $(x, y) = (0,0)$, this spectral localizer is skew-symmetric

So can define the energy-resolved invariant

$$\zeta_E(X, Y, H) = \text{sign}\left[\text{Pf}[L_{(x,y,E)}(WXW^\dagger, WYW^\dagger, WHW^\dagger)]\right]$$

Invariant distinguishes systems based on what atomic limits they can be path continued to

$$\zeta_E \in \{-1,1\} \cong \mathbb{Z}_2$$

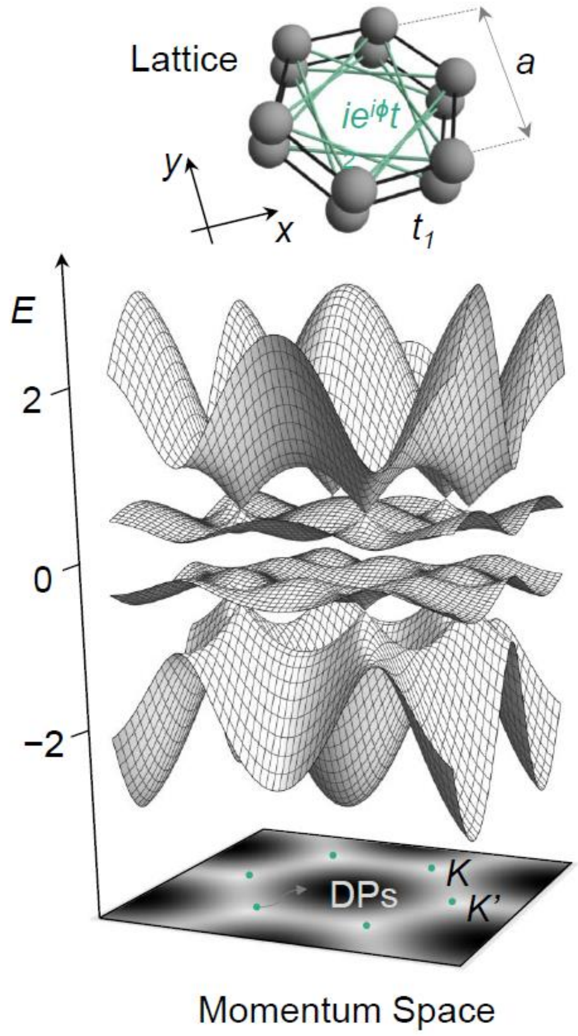
as expected

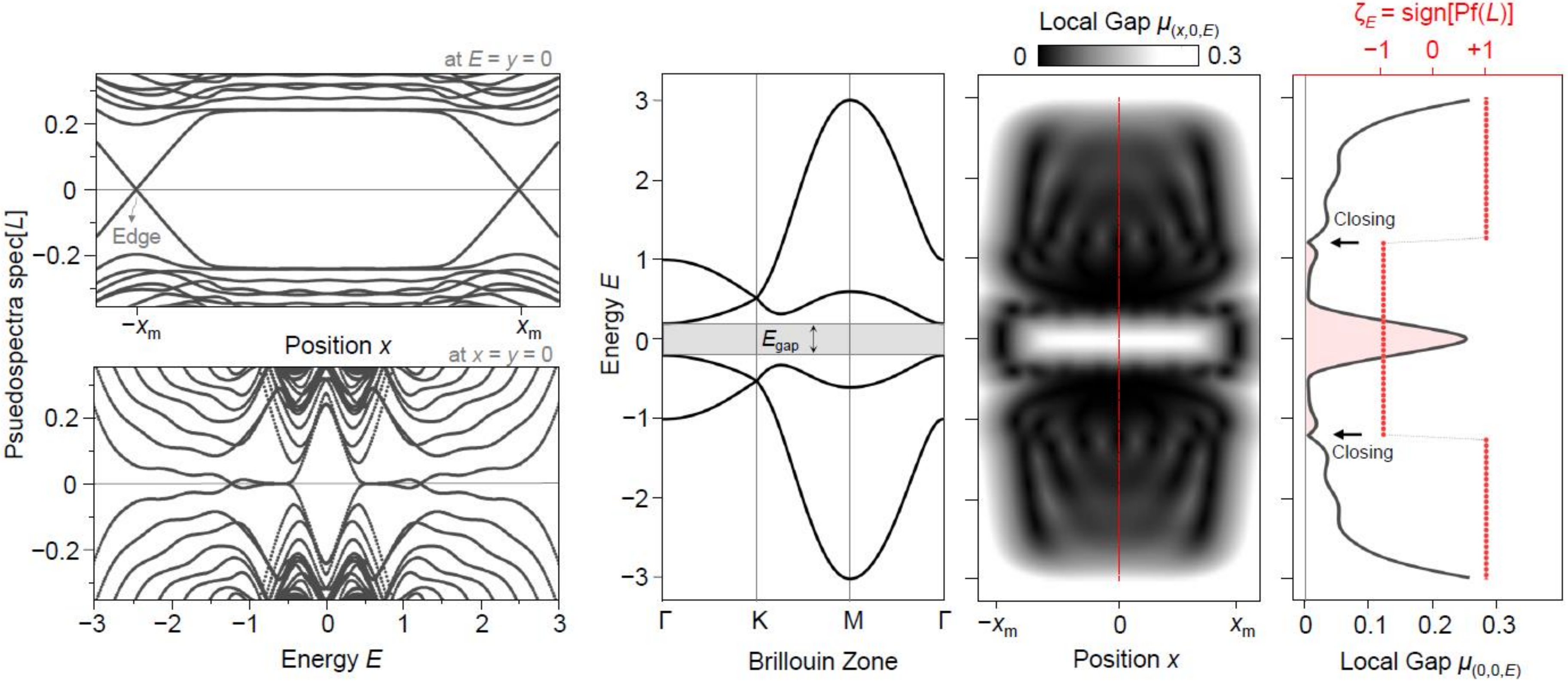
Same definition of topological protection

$$\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)} = \sigma_{\min}[L_{(x_1,\ldots,x_d,E)}(X_1, \ldots, X_d, H)]$$

Lee, Wong, Vaidya, Loring, and AC, arXiv:2503.03948.

Classifying fragile topology via matrix homotopy





Ki Young Lee

Ahn, Park, and Yang, *Phys. Rev. X* **9**, 21013 (2019)

Lee, Wong, Vaidya, Loring, and AC, arXiv:2503.03948.

General framework for non-linear topology

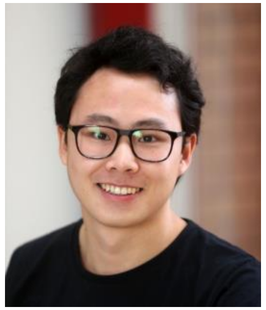
Working in real-space

- Can handle spatial non-linearities **for free**

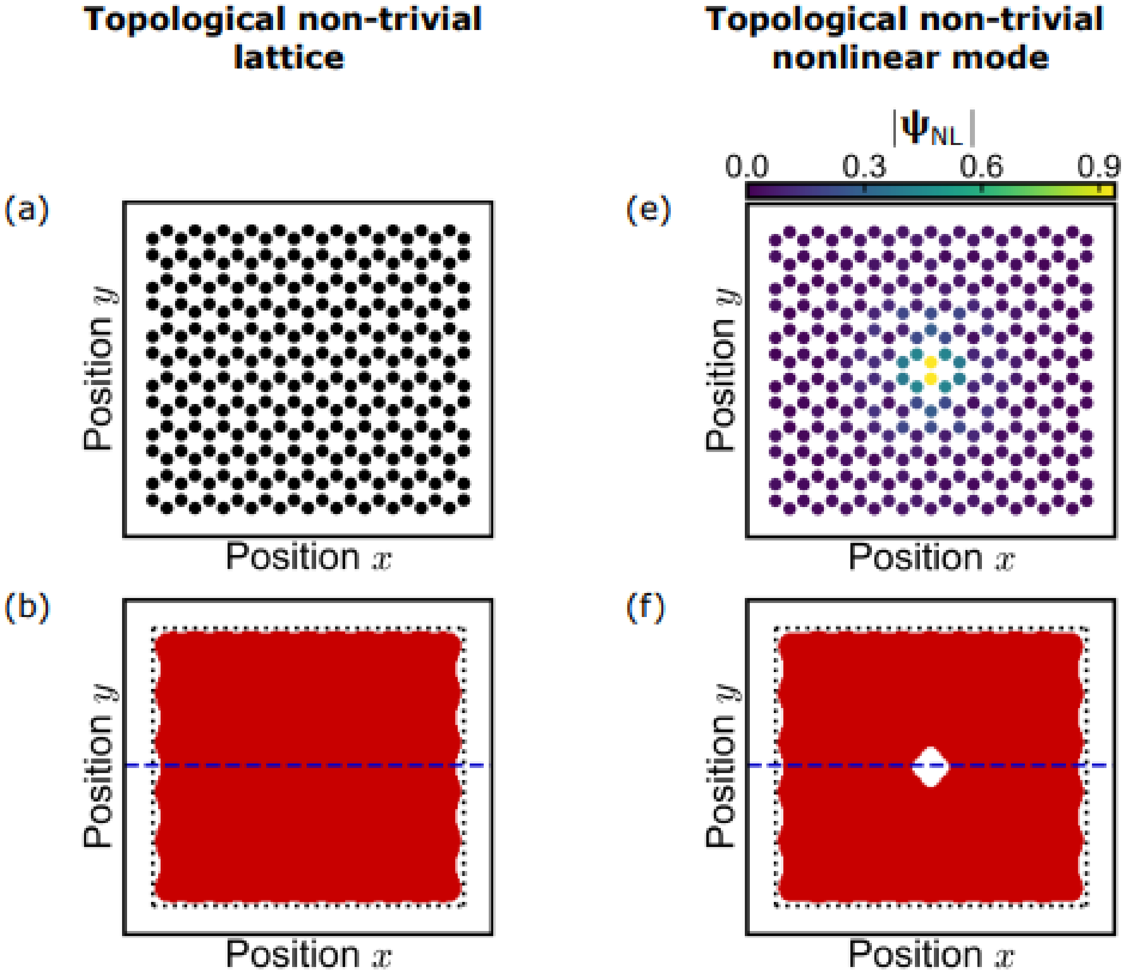
$$L_{(x,y,E)}(X,Y,H_{\mathrm{NL}}(\boldsymbol{\psi})) = \begin{bmatrix} H_{\mathrm{NL}}(\boldsymbol{\psi}) - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H_{\mathrm{NL}}(\boldsymbol{\psi}) - EI) \end{bmatrix}$$

On-site non-linearity

$$H_{\mathrm{NL}}(\boldsymbol{\psi}) = H_0 + g|\boldsymbol{\psi}|^2$$



Stephan Wong



Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

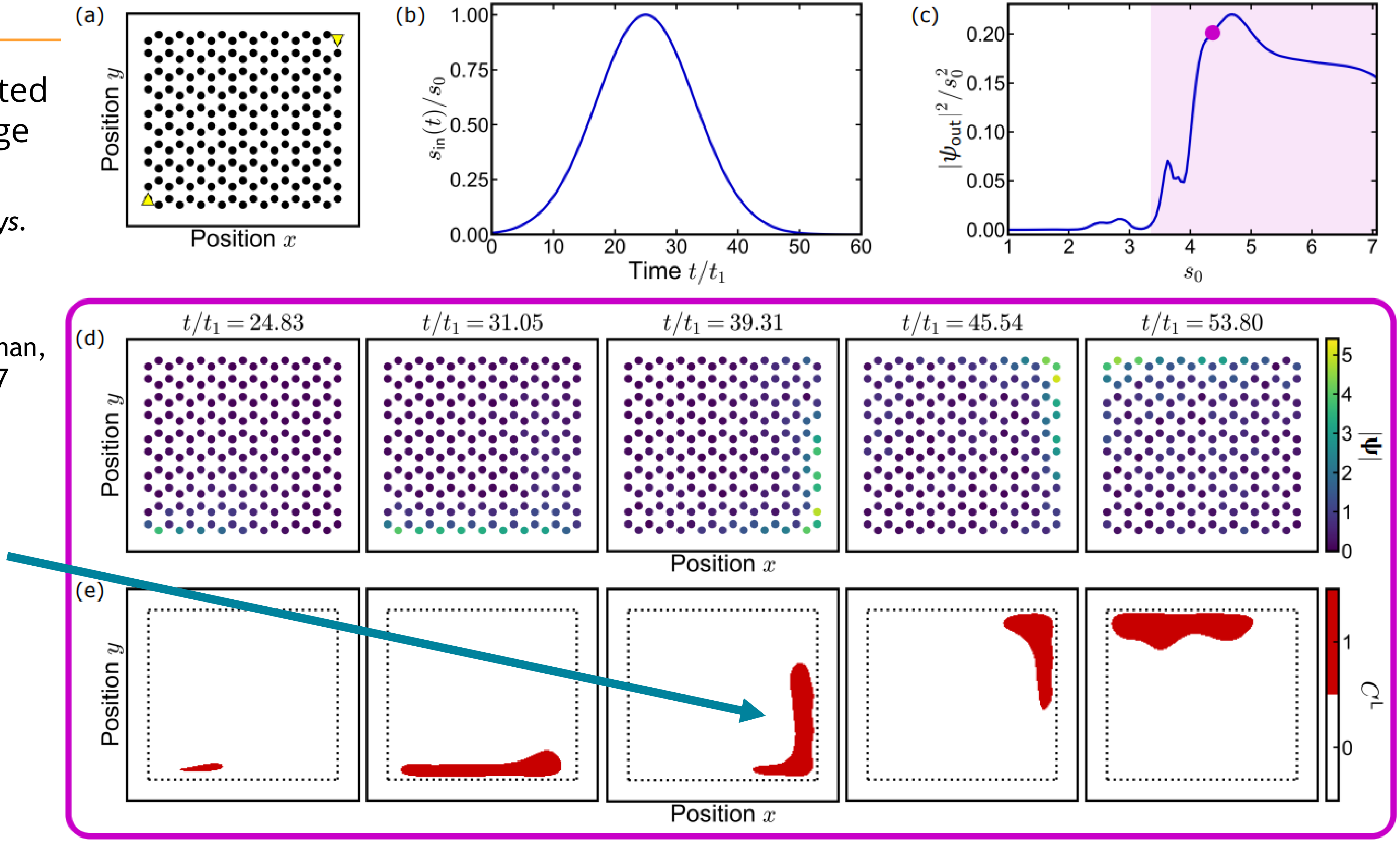
Topological dynamics

Previously predicted and observed edge solitons

Leykam and Chong, *Phys. Rev. Lett.* **117**, 143901 (2016)

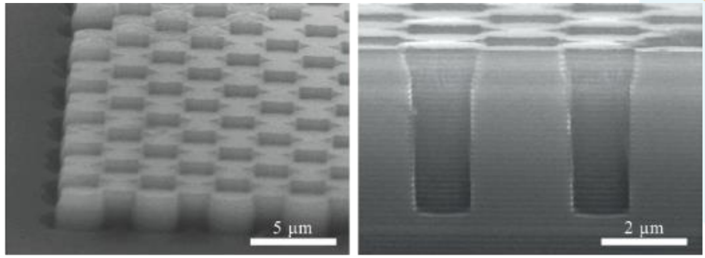
Mukherjee and Rechtsman, *Phys. Rev. X* **11**, 041057 (2021)

Non-linear topological dynamics!



Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

Reconfigurable topology in exciton-polariton lattices



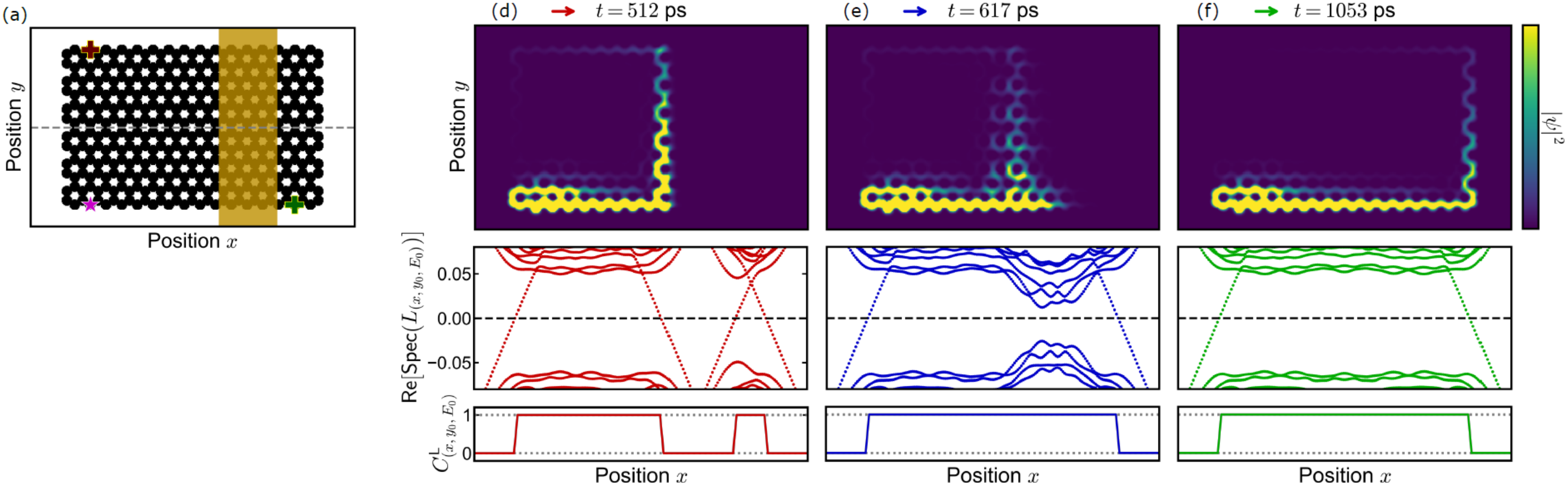
Parameters from

Klembt et al., *Nature* **562**, 552 (2018)

Driven-dissipative exciton-polariton systems

$$i\hbar\frac{\partial}{\partial t}\psi = H_0\psi - i\hbar\left(\frac{\gamma_c}{2}\right)\psi + g_c|\psi|^2\psi + \left(g_r + i\hbar\frac{R}{2}\right)n_r\psi + S_{probe}$$

$$\frac{\partial}{\partial t}n_r = -(\gamma_r + R|\psi|^2)n_r + S_{pump}$$



Wong, Betzold, Höfling, and AC, in submission.

Reformulating Maxwell’s equations

Linear, local media, allow for dispersion

$$\nabla \times \mathbf{E}(\mathbf{x}) = i\omega\bar{\mu}(\mathbf{x},\omega)\mathbf{H}(\mathbf{x})$$
$$\nabla \times \mathbf{H}(\mathbf{x}) = -i\omega\bar{\varepsilon}(\mathbf{x},\omega)\mathbf{E}(\mathbf{x})$$
$$\nabla \cdot [\bar{\varepsilon}(\mathbf{x},\omega)\mathbf{E}(\mathbf{x})] = 0$$
$$\nabla \cdot [\bar{\mu}(\mathbf{x},\omega)\mathbf{H}(\mathbf{x})] = 0$$

For non-zero frequencies, can recast as:

$$\left[\begin{pmatrix} & -i\nabla\times \\ i\nabla\times & \end{pmatrix} - \omega\begin{pmatrix} \bar{\mu}(\mathbf{x},\omega) & \\ & \bar{\varepsilon}(\mathbf{x},\omega)\end{pmatrix}\right]\begin{pmatrix}\mathbf{H}(\mathbf{x}) \\ \mathbf{E}(\mathbf{x})\end{pmatrix} = 0$$

The divergence equations can be recovered using $\nabla \cdot \nabla \times \mathbf{F}(\mathbf{x}) = 0$ for any vector field $\mathbf{F}(\mathbf{x})$, for any $\omega \neq 0$

This yields a “self-consistent” generalized eigenvalue equation:

$$\boxed{W\boldsymbol{\psi}(\mathbf{x}) = \omega M(\mathbf{x},\omega)\boldsymbol{\psi}(\mathbf{x})}$$

$$W = \begin{pmatrix} & -i\nabla\times \\ i\nabla\times & \end{pmatrix} \qquad \boldsymbol{\psi}(\mathbf{x}) = \begin{pmatrix}\mathbf{H}(\mathbf{x}) \\ \mathbf{E}(\mathbf{x})\end{pmatrix}$$

$$M(\mathbf{x},\omega) = \begin{pmatrix} \bar{\mu}(\mathbf{x},\omega) & \\ & \bar{\varepsilon}(\mathbf{x},\omega)\end{pmatrix}$$

And finally an ordinary eigenvalue equation:

$$\boxed{H\boldsymbol{\phi}(\mathbf{x}) = \omega\boldsymbol{\phi}(\mathbf{x})}$$

$$H = M^{-1/2}(\mathbf{x},\omega)WM^{-1/2}(\mathbf{x},\omega)$$

$$\boldsymbol{\phi}(\mathbf{x}) = M^{1/2}(\mathbf{x},\omega)\boldsymbol{\psi}(\mathbf{x})$$

Reformulating Maxwell's equations

By discretizing the system
  - Yee grid
  - Finite-element method

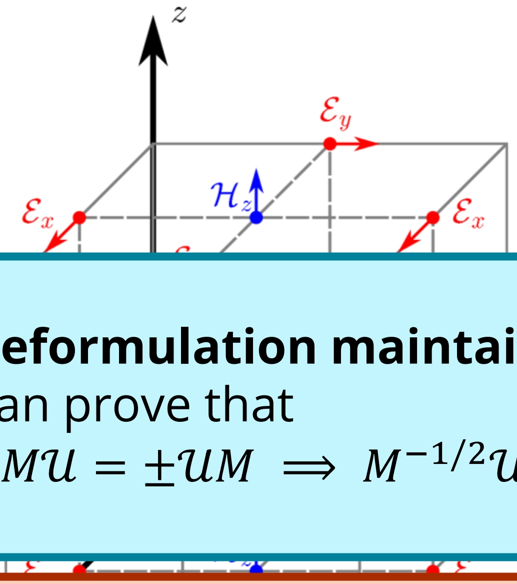
Obtain a lattice, with effective Hamiltonian

$$H_{\text{eff}} = M^{-1/2}(\mathbf{x},\omega) W M^{-1/2}(\mathbf{x},\omega)$$

And the position operators,

$$X, Y, Z$$

are diagonal matrices of the lattice vertex coordinates.



- ❖ **This reformulation maintains symmetries**
  - Can prove that

$$M\mathcal{U} = \pm\mathcal{U}M \implies M^{-1/2}\mathcal{U} = \pm\mathcal{U}M^{-1/2}$$

- ❖ **Numerically, it is impossible to do this for local markers involving projectors**
  - Projectors make sparse matrices dense.

Directly insert into spectral localizer:

$$L_{(x,y,\omega)}(X,Y,H) = \begin{bmatrix} H - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - \omega I) \end{bmatrix}$$

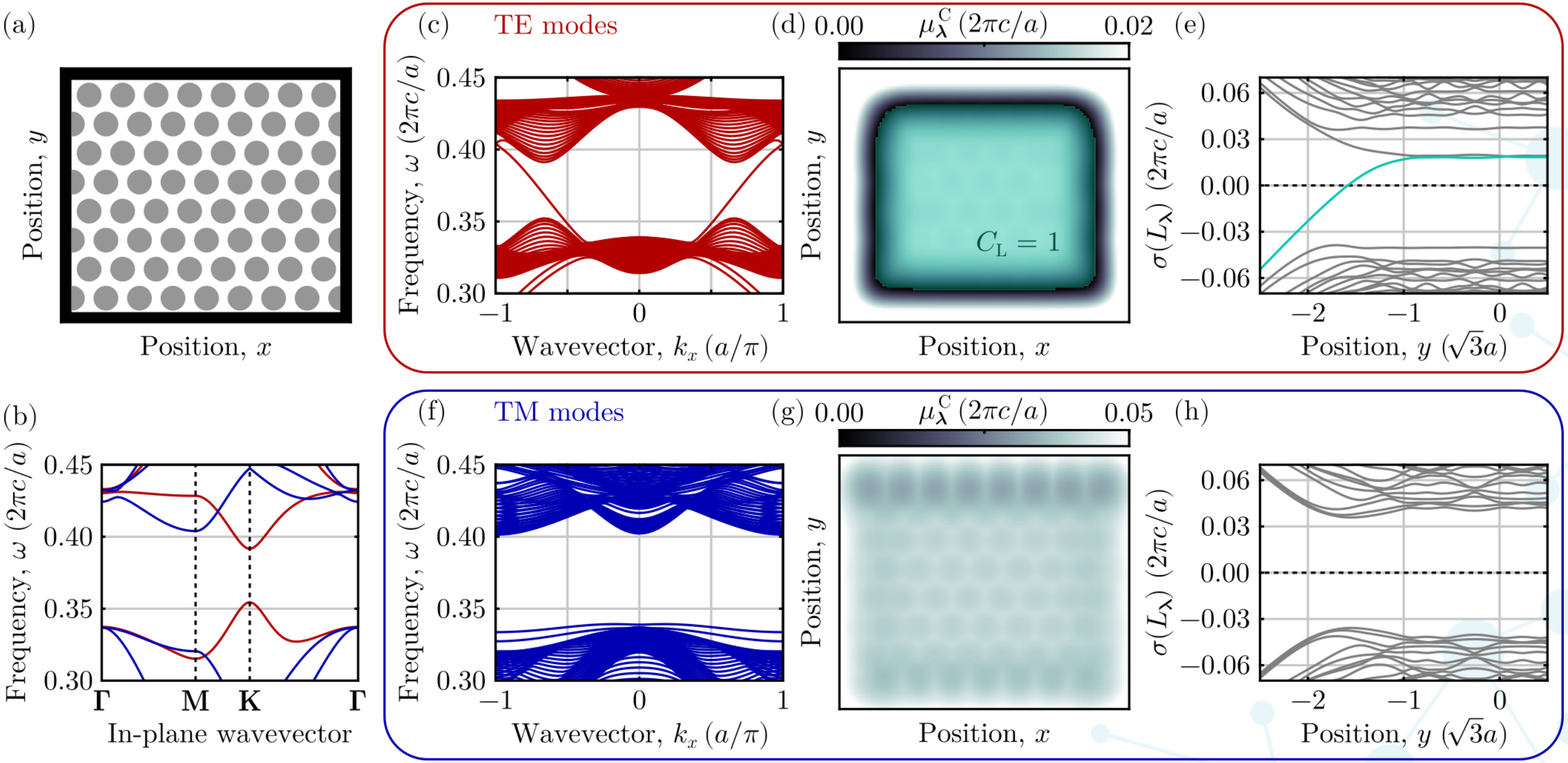
AC and Loring, *Nanophotonics* **11**, 4765 (2022)

The Haldane and Raghu photonic Chern insulator

$$H = M^{-1/2}(\mathbf{x}, \omega) W M^{-1/2}(\mathbf{x}, \omega)$$

$$L_{(x,y,\omega)}(X, Y, H) = \begin{bmatrix} H - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - \omega I) \end{bmatrix}$$

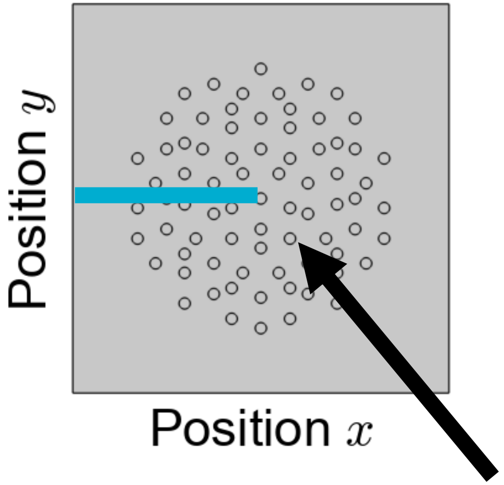
2D photonic crystal of dielectric pillars in gyro-electric air



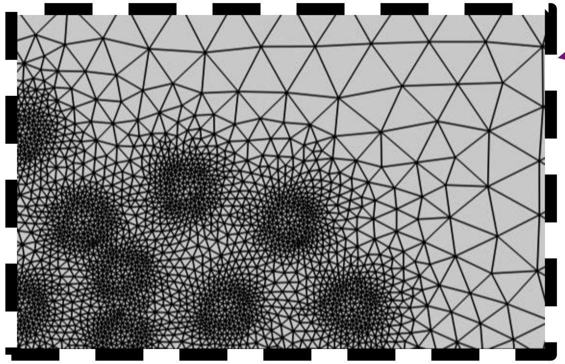
Haldane and Raghu, *Phys. Rev. Lett.* **100**, 013904 (2008)
Raghu and Haldane, *Phys. Rev. A* **78**, 033834 (2008)

AC and Loring, *Nanophotonics* **11**, 4765 (2022)

Photonic Chern Quasicrystal

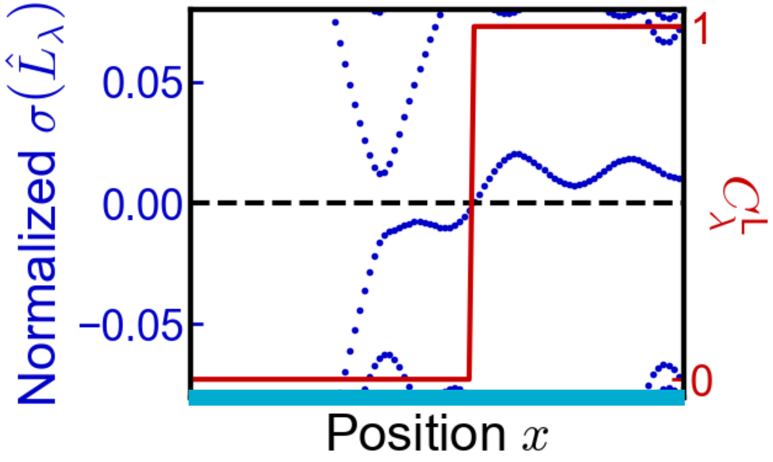


Magneto-optic

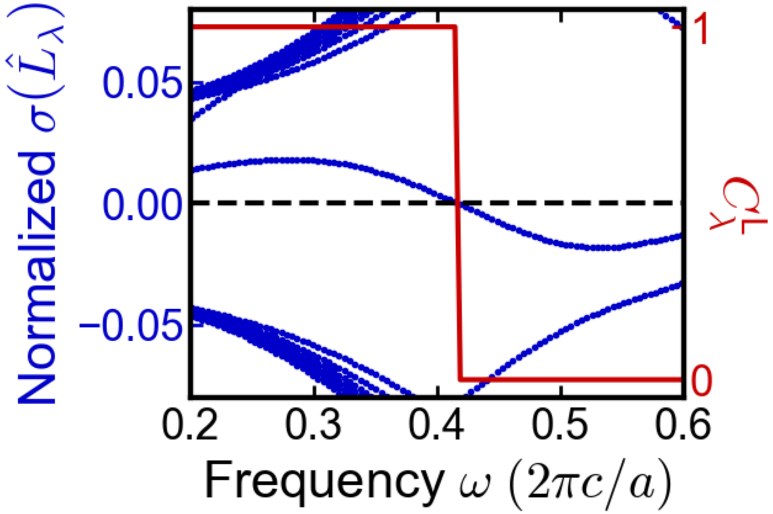


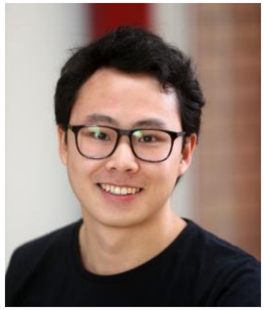
- $L_{\lambda=(x,y,\omega)}(\boxed{X,Y,H_{\text{eff}}})$
- $C^L_\lambda(x,y,\omega) = \frac{1}{2}\text{sig}[L_{(x,y,\omega)}(X,Y,H_{\text{eff}})]$

Vary x, at $\omega = 0.37\ [2\pi c/a]$



Vary ω, at (x_0, y_0)





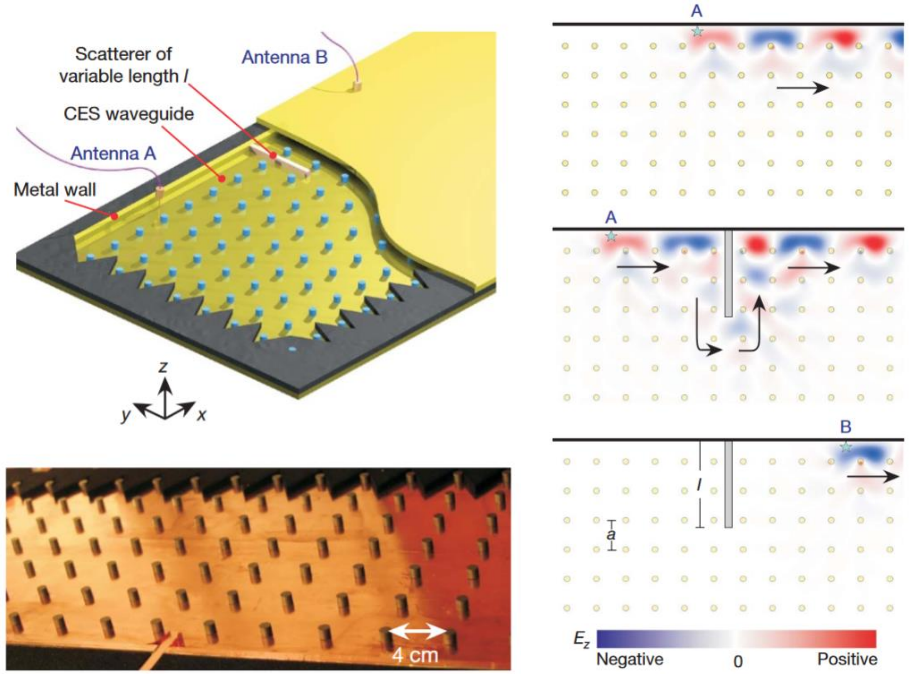
Stephan Wong

Wong, Loring, and AC, *npj Nanophoton.* **1**, 19 (2024)

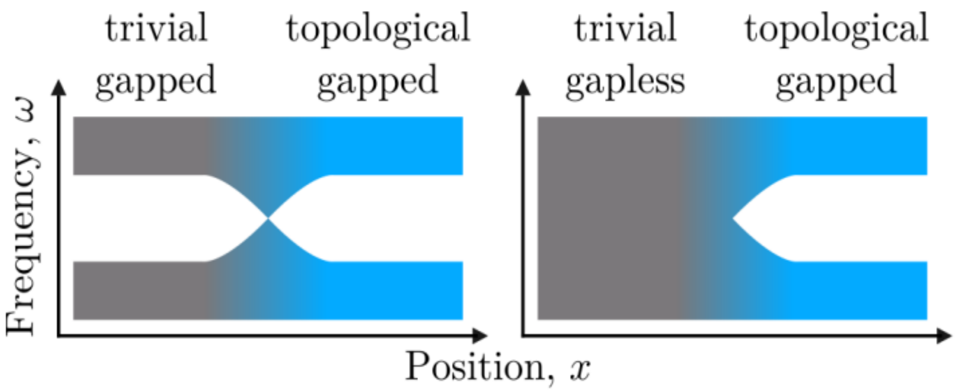
Radiative environments

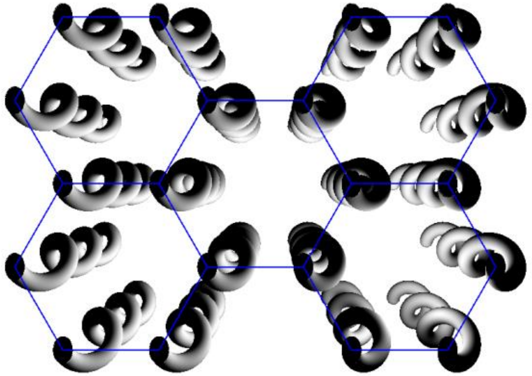
Realized in microwaves

- Surrounded by a metal
  - Acts as perfect electric conductor

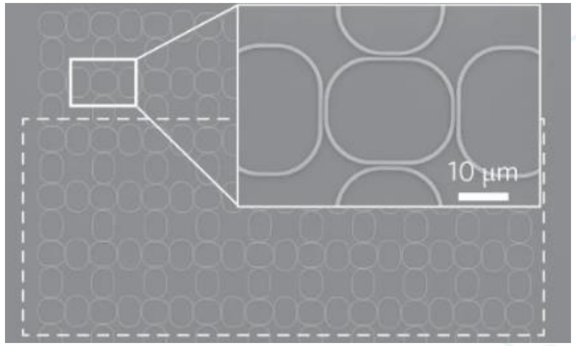


Wang et. al., *Nature* (2009)





Rechtsman et al., *Nature* (2013)



Hafezi et al., *Nat. Photon.* (2013)

Later realizations in other platforms

- Surrounded by air
  - Subject to bending loss

i.e., radiation

Any topological protection against environment perturbations?

Radiative environments

For Chern number, non-Hermitian generalization is known

$$L_{(x,y,\omega)}(X,Y,H) = \begin{bmatrix} H - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - \omega I)^{\dagger} \end{bmatrix}$$

Yielding

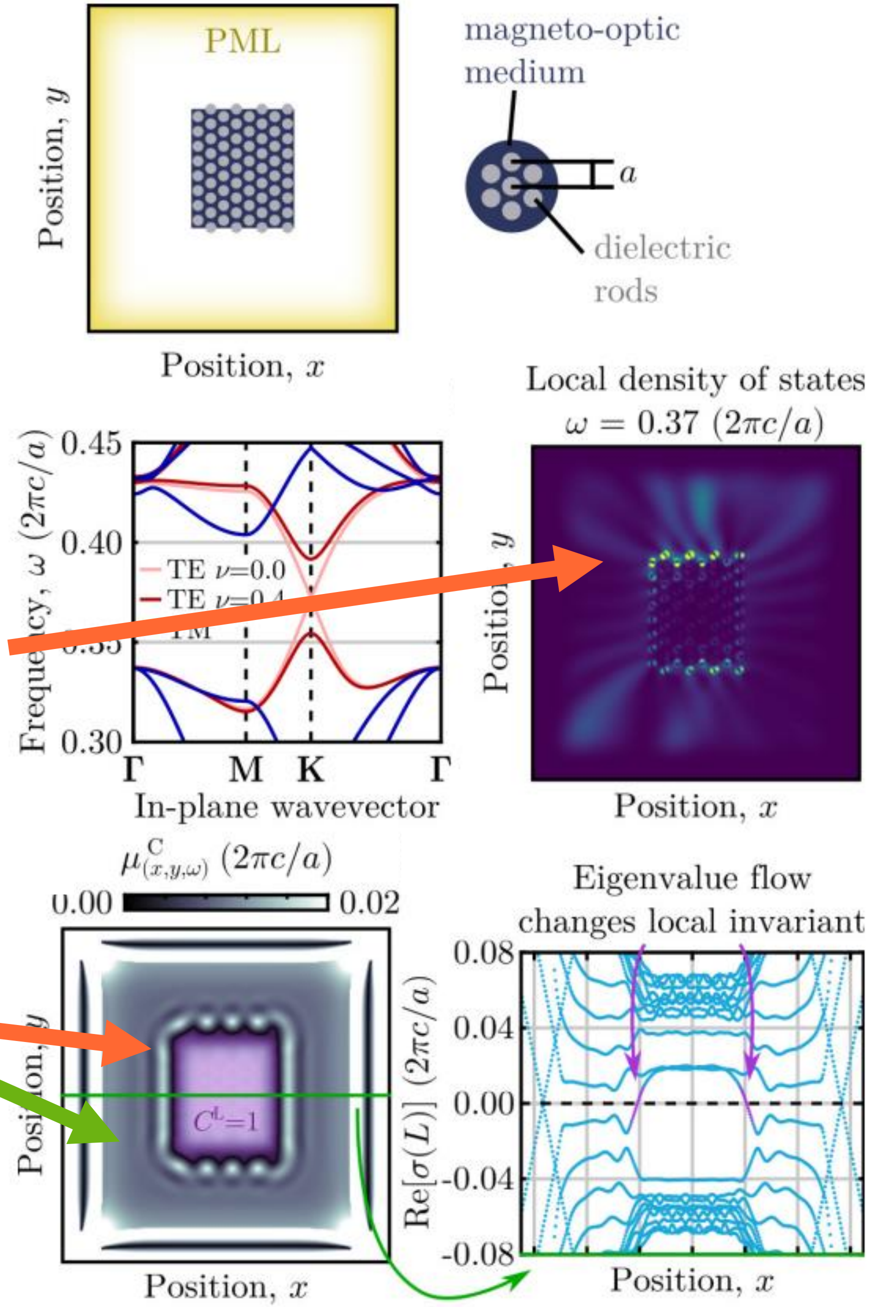
Non-zero local gap!

- ➢ Topological protection against perturbations in the environment!

LDOS shows a chiral edge resonance

Spectral localizer proves existence of chiral edge resonance

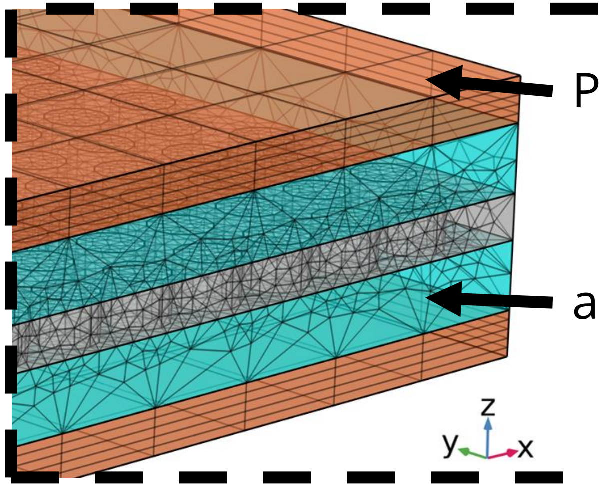
- ➢ Resonance… not state.
- ➢ Couples to vacuum.



Kahlil Y. Dixon

Dixon, Loring, and AC, *Phys. Rev. Lett.* **131**, 213801 (2023)

Topology in Photonic Crystal Slabs



PML: radiative boundary

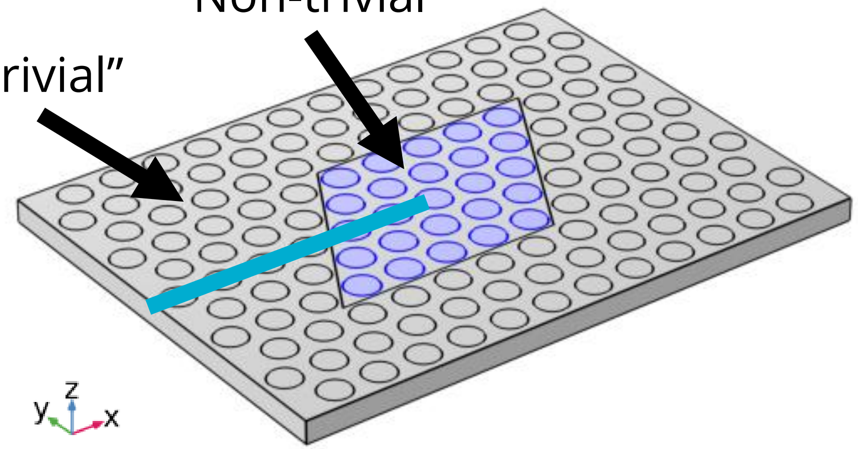
air: gapless environment

class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Schnyder *et. al.*, *Phys. Rev. B* **78**, 195125 (2008)

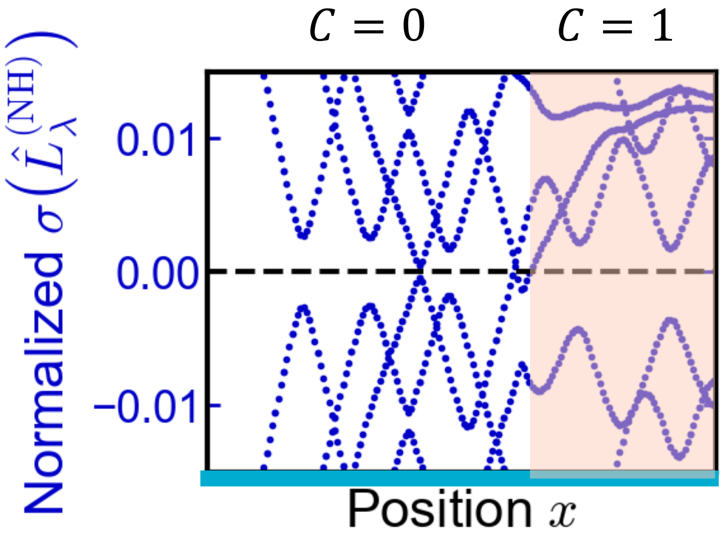
Topological edge states in slab with 2D strong topological invariant

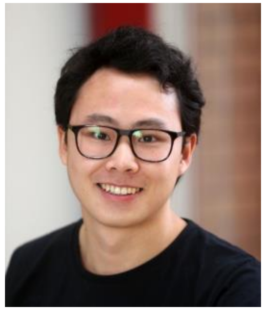
- Disregard z-direction: $(x, y, z) \to (x, y)$
  (still have all vertices, just "forgetting" about z)
- Look at the change of topology in the (x, y)-plane



"Non-trivial"

"Trivial"





Stephan Wong

Wong, Loring, and AC, *npj Nanophoton.* **1**, 19 (2024)

Operators don’t care about physical meaning

In 1D class AIII (e.g., SSH model), chiral symmetry protects states at $E = 0$

$$H\Pi = -\Pi H, \quad X\Pi = \Pi X, \quad \Pi^2 = I, \quad \Pi = \Pi^\dagger$$

Local winding number:

$$\upsilon_{(x,0)} = \frac{1}{2}\mathrm{sig}[(\kappa(X - xI) + iH)\Pi] \in \mathbb{Z}$$

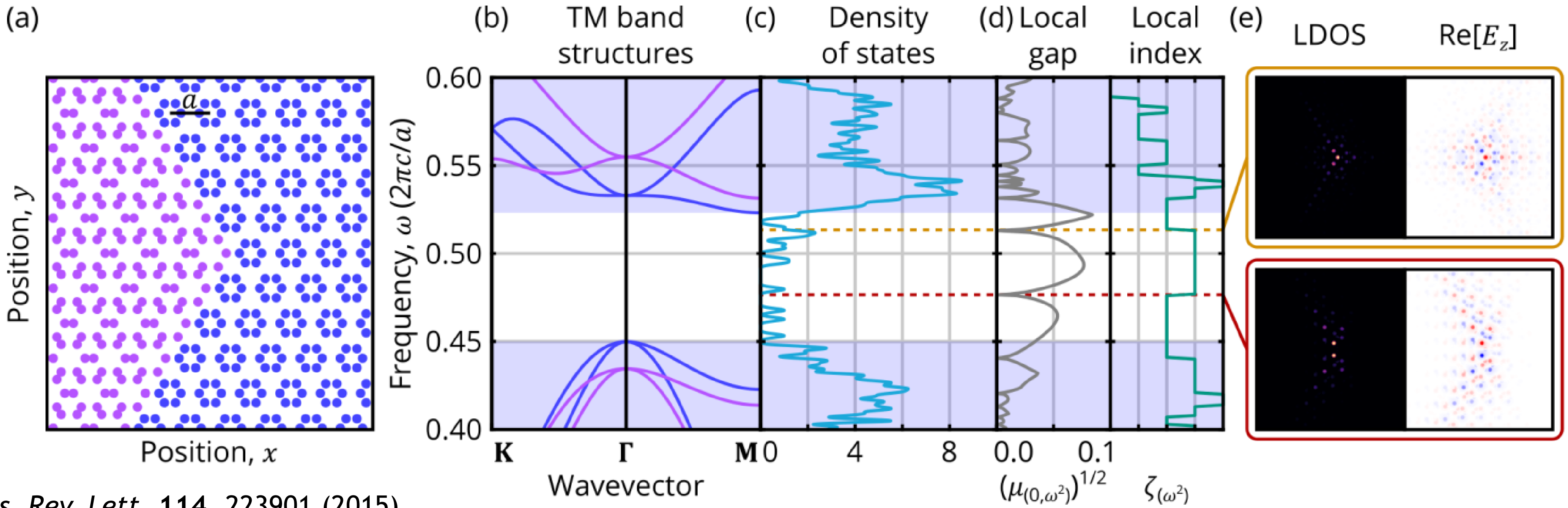
But crystalline symmetry can yield similar commutation relations

$$H\mathcal{S} = \mathcal{S}H, \quad X\mathcal{S} = -\mathcal{S}X, \quad \mathcal{S}^2 = I, \quad \mathcal{S} = \mathcal{S}^\dagger$$

Local “crystalline winding number,” protects states at $x = 0$

$$\zeta^{\mathcal{S}}_{(0,E)} = \frac{1}{2}\mathrm{sig}[(H - EI + i\kappa X)\mathcal{S}] \in \mathbb{Z}$$

Local markers for crystalline topology



Wu, Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Smirnova et al., *Phys. Rev. Lett.*, **123**, 103901 (2019)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Local “reflection winding number,” protects states at $y = 0$

$$\zeta^{\mathcal{R}_y}_{(0,\omega^2)} = \frac{1}{2}\operatorname{sig}\left[(H - \omega I + i\kappa Y)\mathcal{R}_y\right] \in \mathbb{Z}$$

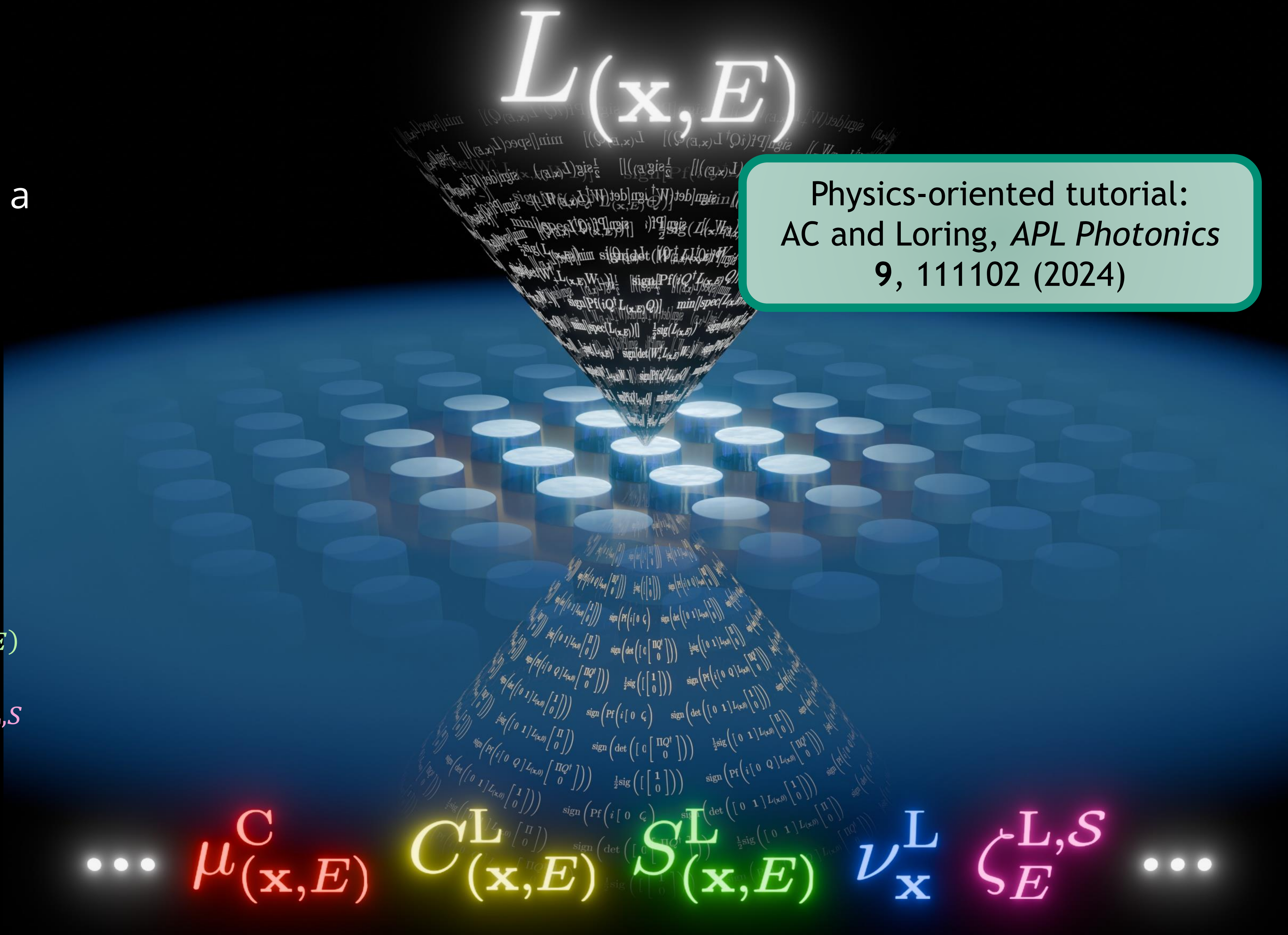
AC, Loring, and Schulz-Baldes, *Phys. Rev. Lett.* **132**, 073803 (2024)

A “mathematical SEM”

If $\mu^{\mathrm{C}}_{(\mathbf{x},E)}$ is small – system has a nearby state

If $\mu^{\mathrm{C}}_{(\mathbf{x},E)}$ is large – the local topological phase is robust

- Can be classified with
    - Chern number $C^{\mathrm{L}}_{(\mathbf{x},E)}$
    - Quantum spin Hall $S^{\mathrm{L}}_{(\mathbf{x},E)}$
    - Winding number $\nu^{\mathrm{L}}_{\mathbf{x}}$
    - Crystalline topology $\zeta^{\mathrm{L},S}_{E}$
    - etc…



Physics-oriented tutorial:
AC and Loring, *APL Photonics* **9**, 111102 (2024)