Classifying Topology in Open (and nonlinear) Photonic Systems

Alexander Cerjan

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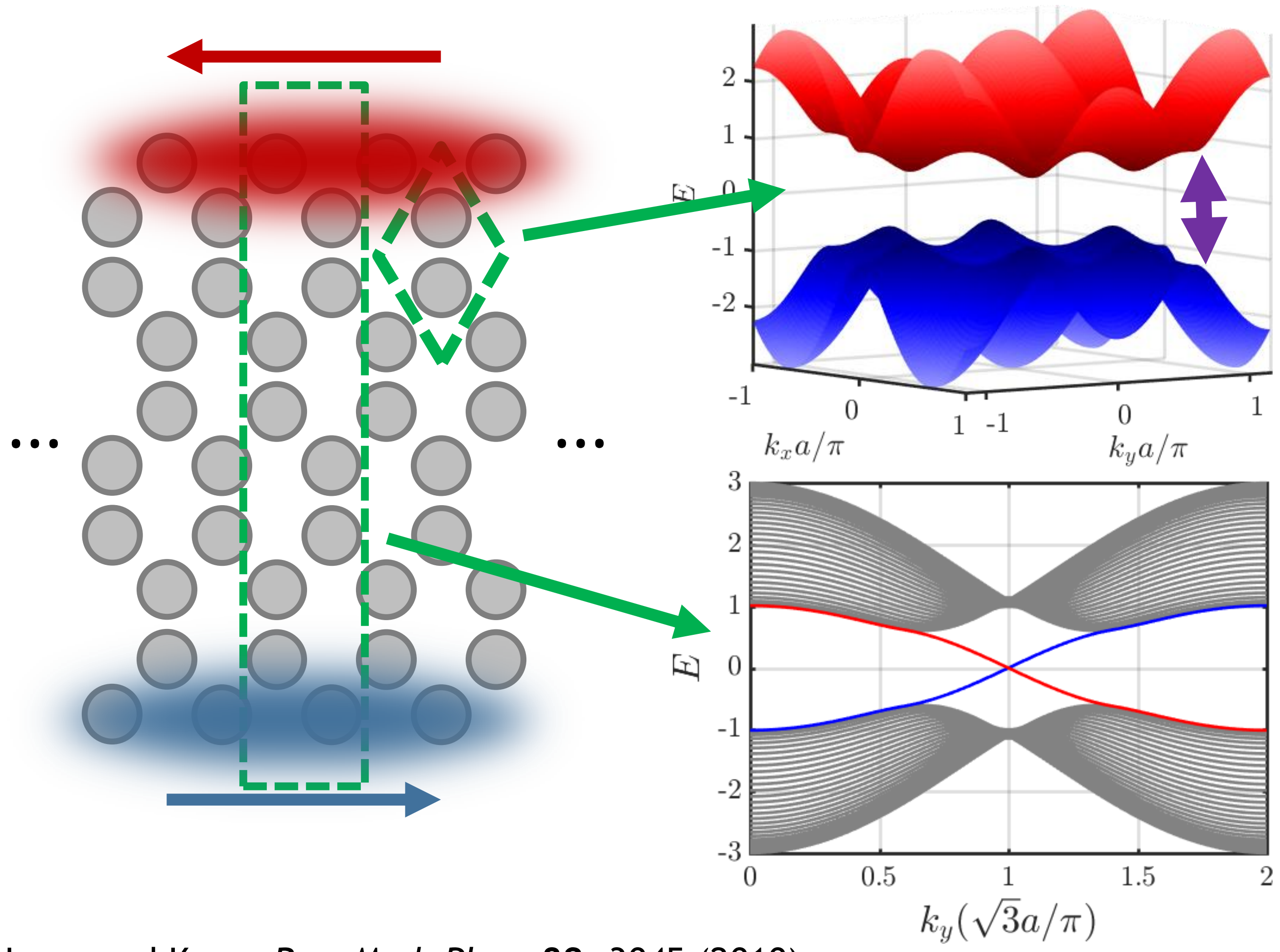
Topology from invariants

Review: Chern insulators

2D system with *broken* Time Reversal symmetry



Can predict these boundary phenomena from a calculation in the bulk

- Bulk-boundary correspondence

Chern number: (a "topological invariant")

$$C_n = \frac{1}{2\pi i}\int_{BZ}\left(\frac{\partial A_y^n}{\partial k_x} - \frac{\partial A_x^n}{\partial k_y}\right) d^2\mathbf{k} \in \mathbb{Z}$$

Berry Connection:

$$\mathbf{A}^n(\mathbf{k}) = i\langle\psi_{n\mathbf{k}}|\nabla_{\mathbf{k}}|\psi_{n\mathbf{k}}\rangle$$

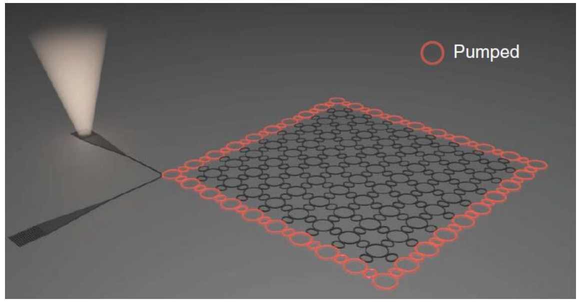
Bloch eigenstates

Hasan and Kane, *Rev. Mod. Phys.* **82**, 3045 (2010)

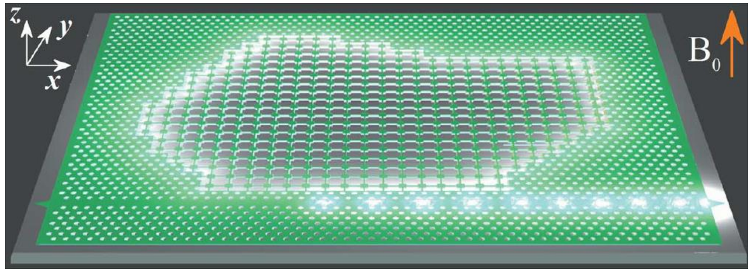
Why make photonics topological?

Topological lasers

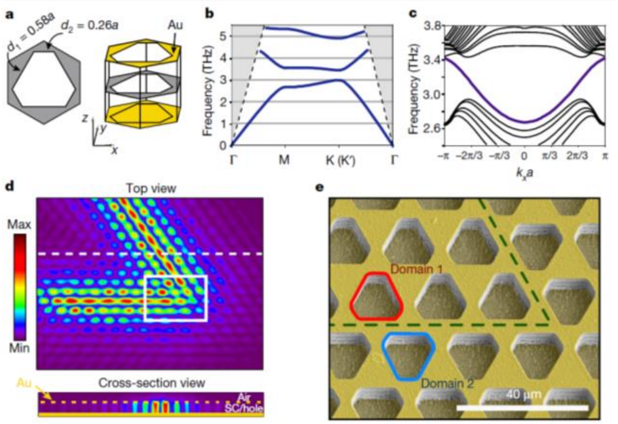
- Robust against disorder
- Efficient phase locking



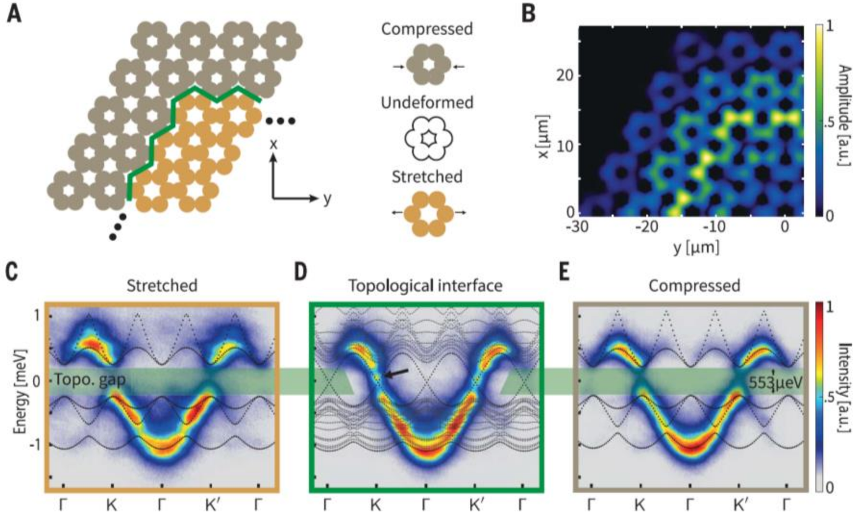
Bandres et al., *Science* **359**, 1231 (2018)
Harari et al., *Science* **359**, eaar4003 (2018)



Bahari et al., *Science* **358**, 636 (2017)



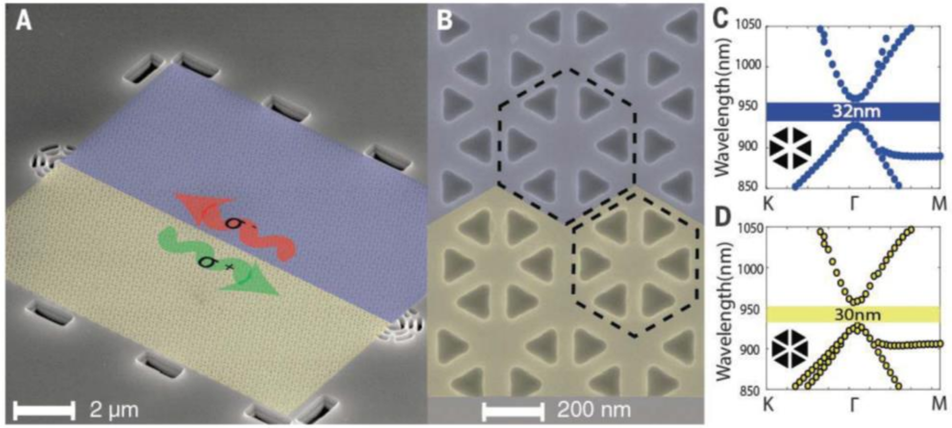
Zeng et al., *Nature* **578**, 246 (2020)



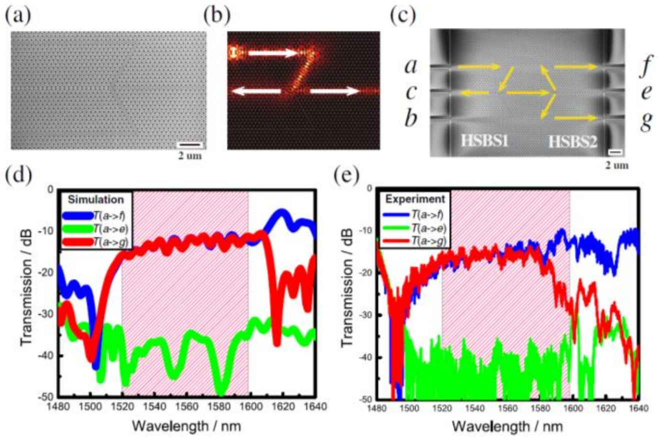
Dikopoltsev et al., *Science* **373**, 1514 (2021)

Why make photonics topological?

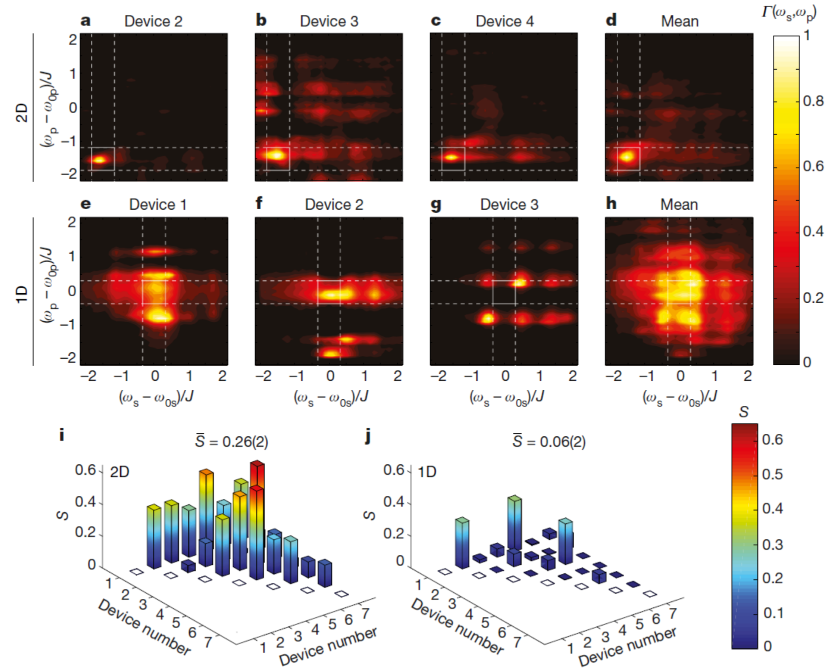
Routing of quantum information



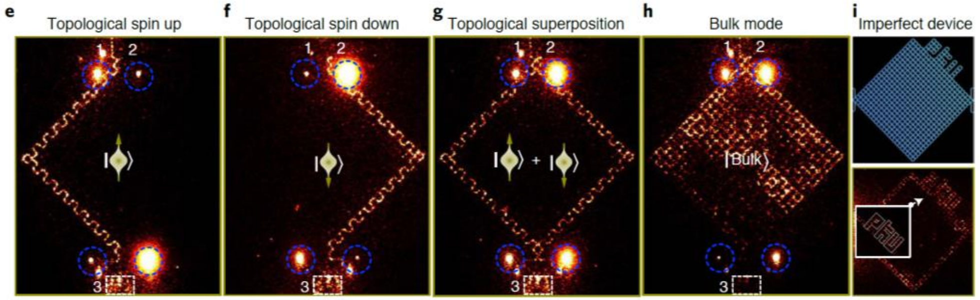
Barik et al., *Science* **359**, 666 (2018)



Chen et al., *Phys. Rev. Lett.* **126**, 230503 (2021)



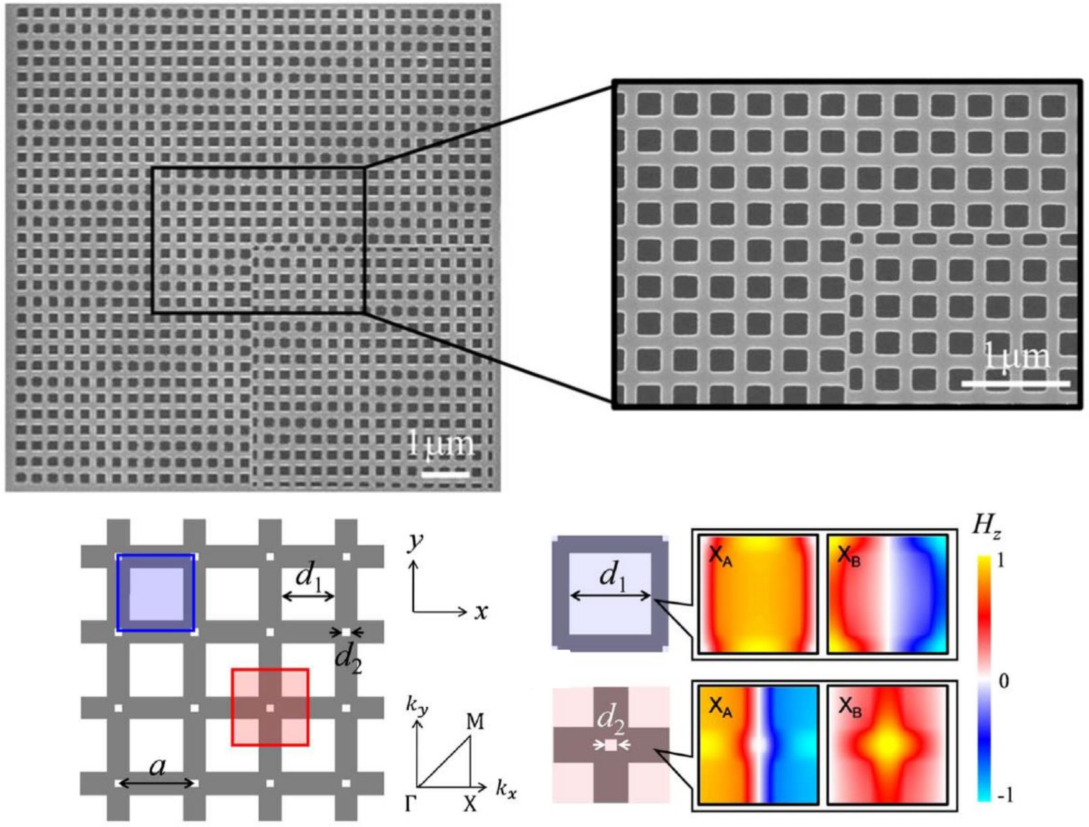
Mittal et al., *Nature* **561**, 502 (2018)



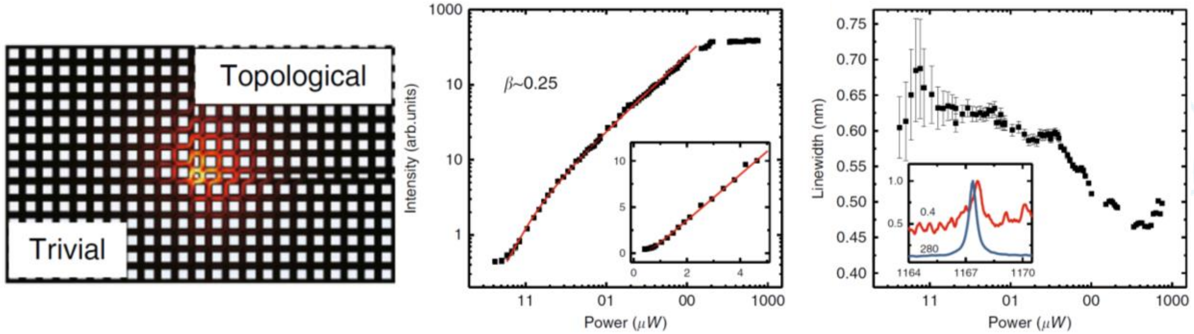
Dai et al., *Nat. Photonics* **16**, 248 (2022)

Why make photonics topological?

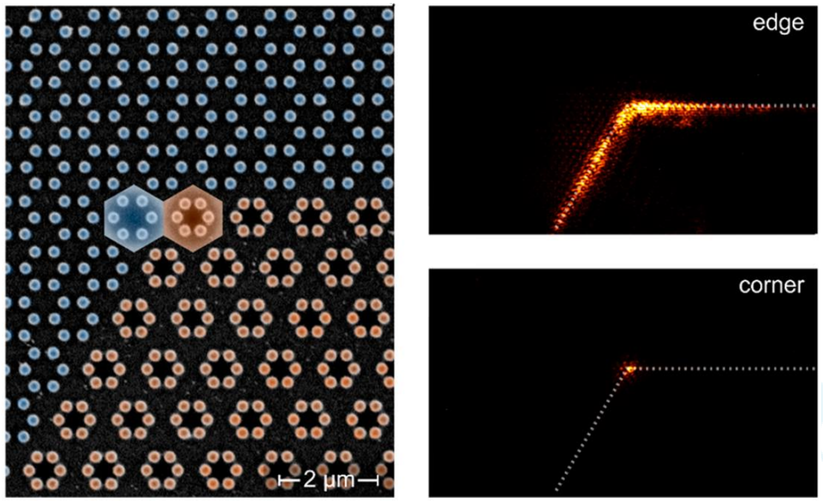
Creating cavities for light-matter interaction



Ota et al., *Optica* **6**, 786 (2019)



Zhang et al., *Light Sci. Appl.* **9**, 109 (2020)



Smirnova et al., *Phys. Rev. Lett.*, **123**, 103901 (2019)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Challenges with invariants

Where band theory can be applied, band theory is awesome.

Where is band theory not applicable (or not useful)?

1) Material lacks translational symmetry
   - Quasicrystals
   - Amorphous materials
   - Disorder
   - Finite size effects

2) Heterostructure lacks a complete or incomplete band gap
   - ➢ Band theory is applicable, but…
     - ➢ Not always clear how to calculate the invariant
     - ➢ No measure of protection

3) System is non-linear
   - Localized response breaks translational symmetry

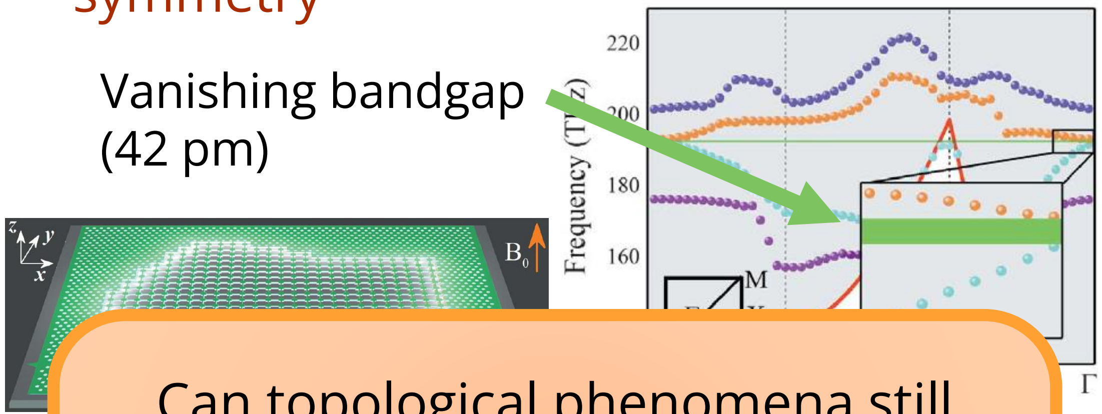
Challenges with invariants in photonics

We'd like nanophotonic Chern insulators

- Non-reciprocal edge states

But… it's hard to break time-reversal symmetry

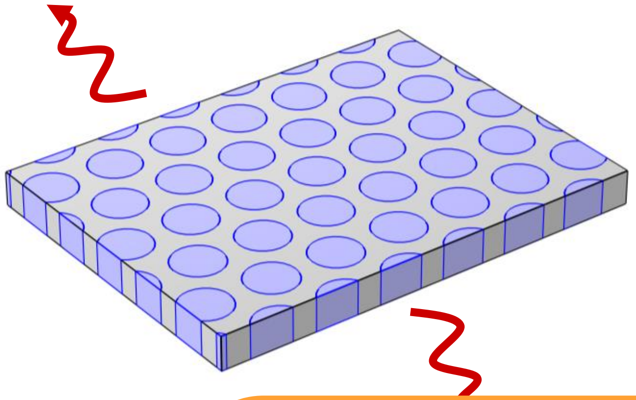
Vanishing bandgap (42 pm)

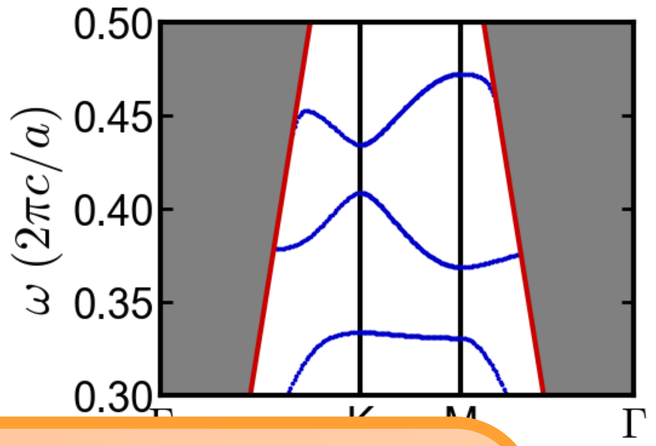


Can topological phenomena still manifest without a complete band gap?

- Chiral edge resonance?

Related challenge: photonic crystal slabs and metasurfaces radiate out-of-plane





Can resonances and bound states be mixed in formula for topological invariants?

$$C_n = \frac{1}{2\pi}\int_{BZ} \nabla_{\mathbf{k}} \times i\,\langle \psi_{n\mathbf{k}} | \nabla_{\mathbf{k}} | \psi_{n\mathbf{k}} \rangle d^2\mathbf{k}$$

Challenges with invariants in photonics

No current theory for finite systems

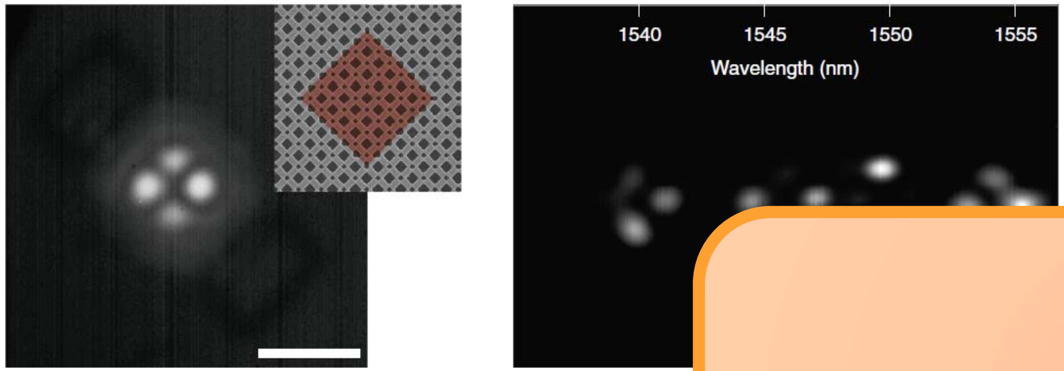
How close can two topological cavities be, while maintaining protection?

Or how close can two chiral edge states be in a topological Chern system?

Estimate:

$$e^{-\frac{x}{L}}$$

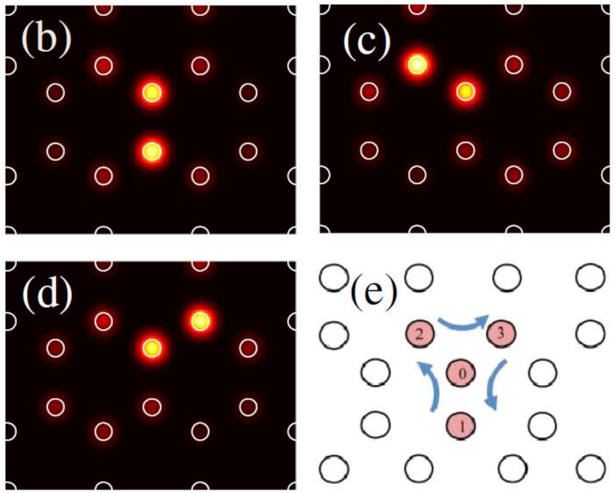
Decay length L set by band gap width ΔE



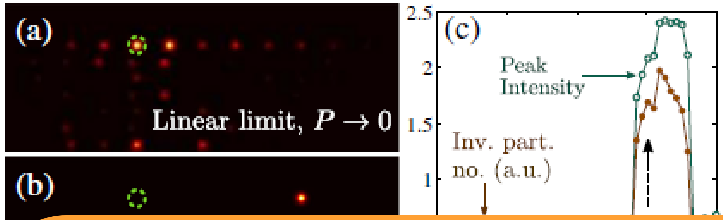
Kim et al., *Nat. Commun.* **11**,

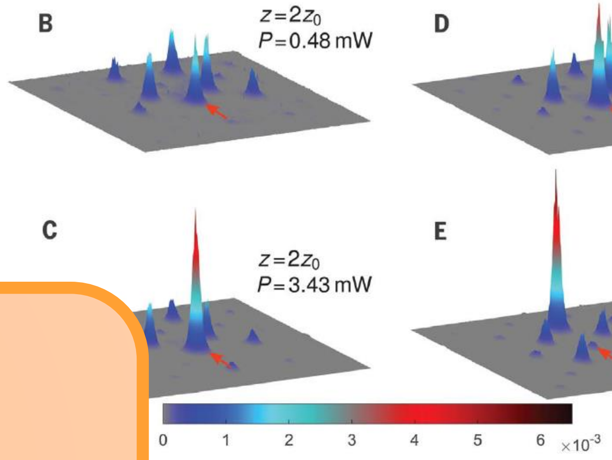
Is there a local measure of topological protection?

Photonic non-linearities are local



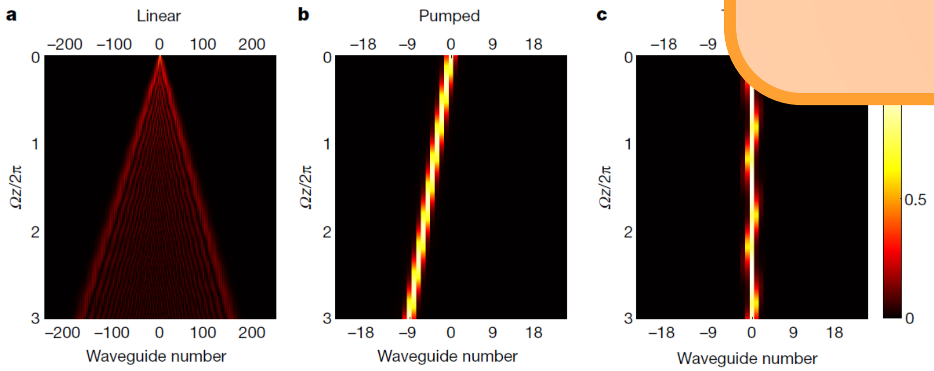
Lumer et al., *Phys. Rev. Lett.* **111**, 243905 (2013)



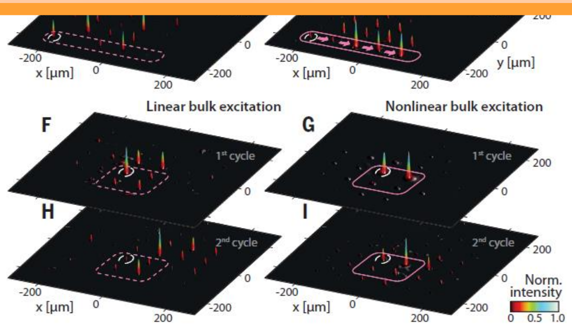


…and Rechtsman, *Science* **368**, 856 (2020)

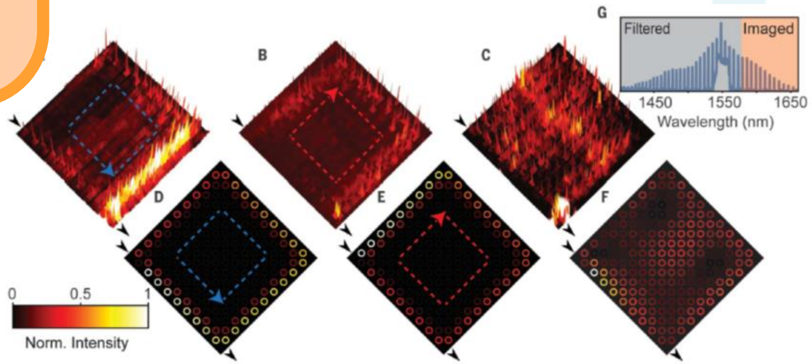
Can a topological invariant be defined without a bulk?



Jürgensen et al., *Nature* **596**, 63 (2021)
Jürgensen et al., *Nat. Phys.* **19**, 420 (2023)

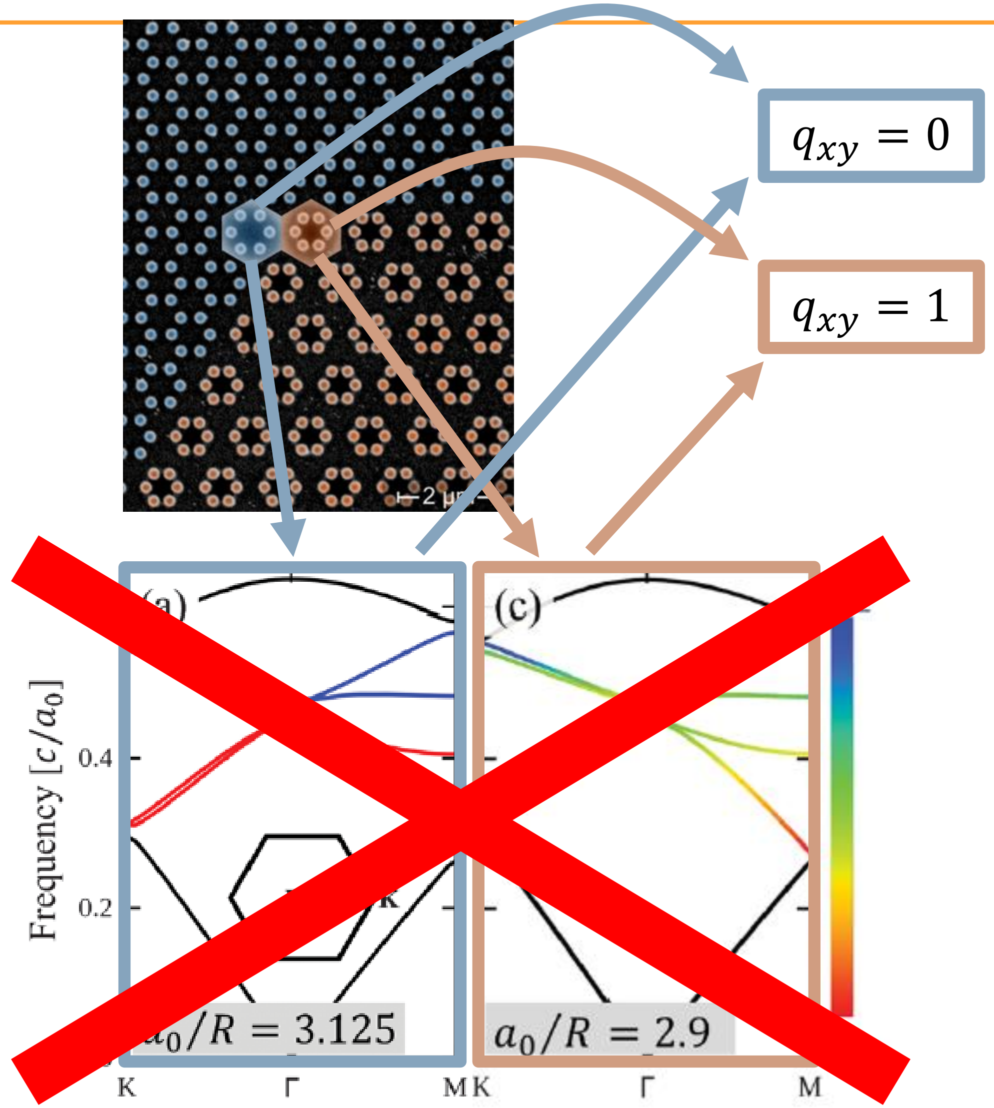


Maczewsky et al., *Science* **370**, 701 (2020)



Flower et al., *Science* **384**, 1356 (2024)

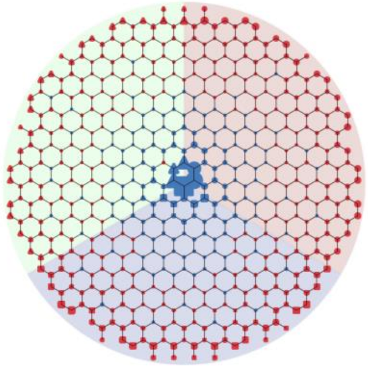
Local real-space approaches to material topology



Wu and Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Kitaev:

$$\nu(P) = 12\pi i \sum_{j\in A}\sum_{k\in B}\sum_{l\in C}\left(P_{jk}P_{kl}P_{lj} - P_{jl}P_{lk}P_{kj}\right)$$



Kitaev, *Ann. Phys.* **321**, 2 (2006)
Mitchell et al., *Nat. Phys.* **14**, 380 (2018)

Bianco–Resta:

$$\mathfrak{C}(\mathbf{r}) = -2\pi i \int [\tilde{X}(\mathbf{r},\mathbf{r}')\tilde{Y}(\mathbf{r}',\mathbf{r}) - \tilde{Y}(\mathbf{r},\mathbf{r}')\tilde{X}(\mathbf{r}',\mathbf{r})]\, d\mathbf{r}'$$

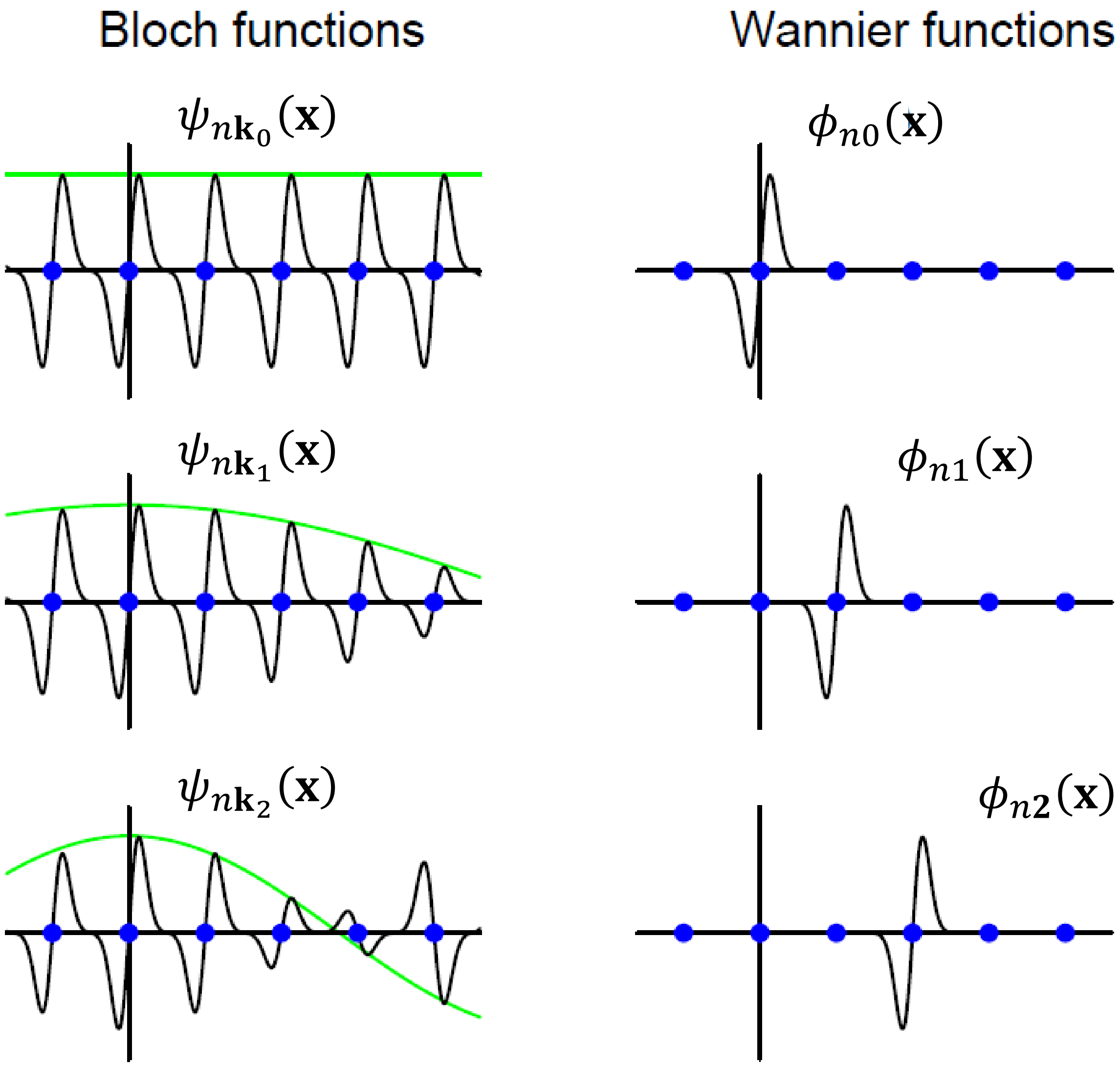
$$\tilde{X}(\mathbf{r},\mathbf{r}') = \int P(\mathbf{r},\mathbf{r}'')x''P(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''$$

Bianco and Resta, *Phys. Rev. B* **84**, 241106(R) (2011)

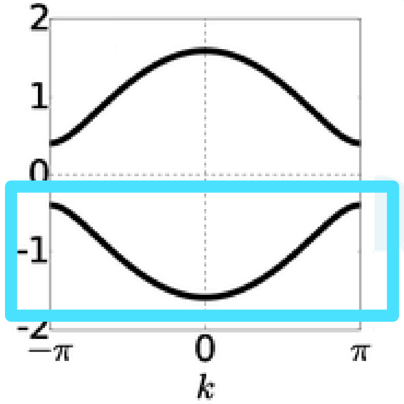
Outline

- An operator-based approach to topological physics
  - Uses a framework called the "*spectral localizer*"
- Topology without a band gap
  - Realization in an acoustic metamaterial
- Classifying topology in non-linear systems
- Application directly to Maxwell's equations
  - Incorporating radiative boundaries
- Application to 2D electron gasses, emergence of Hofstader's butterfly

What is a Wannier basis? (and why should you care?)



Fourier transform of a band with a gauge, i.e., $\theta(\mathbf{k})$



$$\phi_{n\mathbf{R}}(\mathbf{x}) = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\theta(\mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{x}} \psi_{n\mathbf{k}}(\mathbf{x})$$

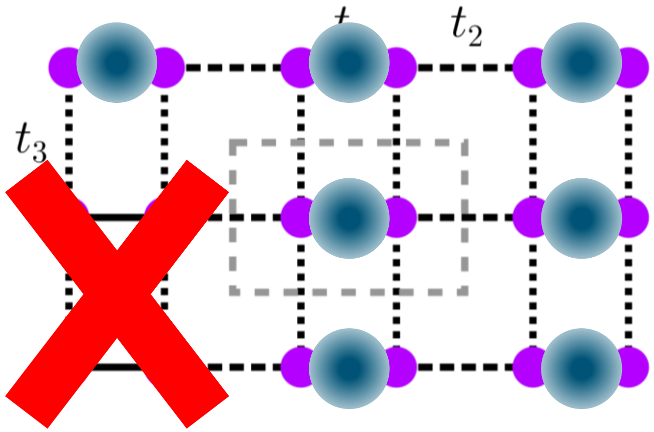
$$\psi_{n\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u_{n\mathbf{k}}(\mathbf{x})$$

Marzari et al., *Rev. Mod. Phys.* **84**, 1419 (2012)

Implications of topology on the Wannier basis

Systems with non-trivial Chern numbers DO NOT possess a complete localized Wannier basis.

$$C_n = \frac{1}{2\pi i}\int_{BZ}\left(\frac{\partial A_y^n}{\partial k_x} - \frac{\partial A_x^n}{\partial k_y}\right) d^2\mathbf{k} \neq 0 \quad \Longleftrightarrow$$



This is an *if and only if* statement

- ➢ No complete localized Wannier basis necessitates a non-trivial Chern number.

This generalizes to many other classes of topology

Example: No localized Wannier basis that respects time-reversal symmetry
$\Leftrightarrow$ non-trivial Kane-Mele invariant (Quantum spin Hall)

Brouder et al., *Phys. Rev. Lett.* **98**, 046402 (2007)

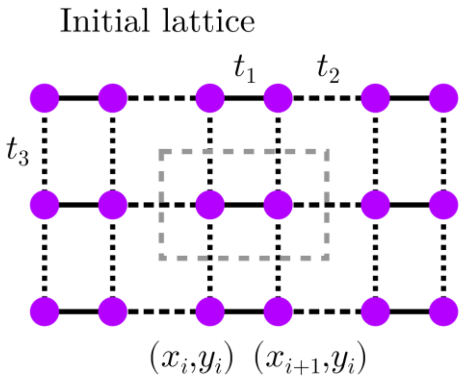
Soluyanov and Vanderbilt, *Phys. Rev. B* **83**, 035108 (2011)

Topology as “Wannierizability”

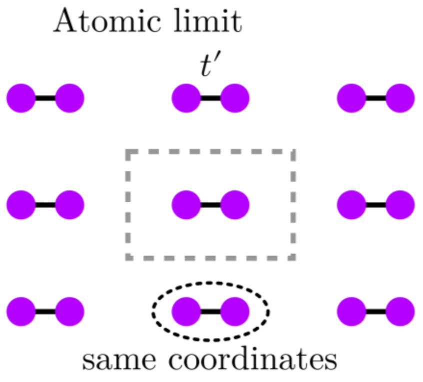
Instead of an invariant, “Does the system possess a complete Wannier basis?”

- Band gap stays open
- Symmetries are preserved

Can a lattice



be continued to



without violating?

If yes

➢ Trivial

If no

➢ Topological

In other words, “Can the system be permuted to an *atomic limit*?”

(and if multiple inequivalent limits exist, which one?)

➢ Can answer using a lattice’s band structure

➢ Topological quantum chemistry

Bradlyn et al., *Nature* **547**, 298 (2017)

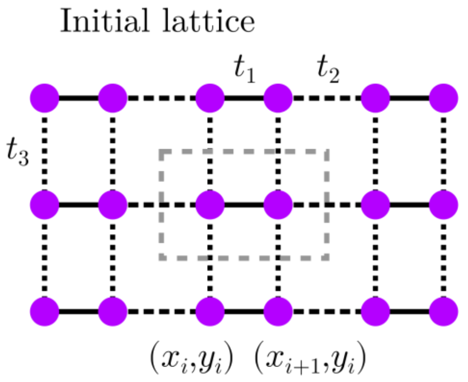
Kitaev, *AIP Conference Proceedings* **1134**, 22 (2009)
Hastings and Loring, *Ann. Phys.* **326** 1699 (2011)
Taherinejad et al., *Phys. Rev. B* **89**, 115102 (2014)
Kruthoff et al., *Phys. Rev. X* **7**, 041069 (2017)
Po et al., *Nat. Commun.* **8**, 50 (2017)

Topology as an atomic limit

Instead of an invariant, “Can the system be permuted to an *atomic limit*?”

- Band gap stays open
- Symmetries are preserved

Can a lattice



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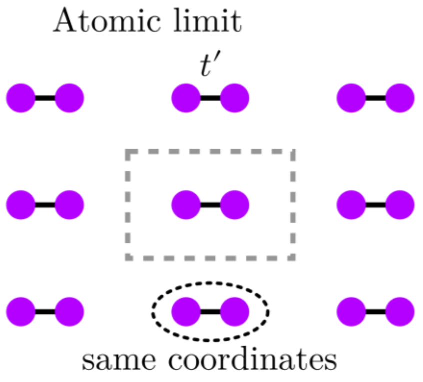


without violating?

If yes
➢ Trivial

If no
➢ Topological

Can the operators

$$H = \begin{bmatrix} \ddots & -t_2 & & -t_3 & & \\ -t_2 & \varepsilon & -t_1 & & -t_3 & \\ & -t_1 & \varepsilon & -t_2 & & -t_3 \\ -t_3 & & -t_2 & \varepsilon & -t_1 & \\ & -t_3 & & -t_1 & \varepsilon & -t_2 \\ & & -t_3 & & -t_2 & \ddots \end{bmatrix}$$

$$X = \begin{bmatrix} \ddots & & & & & \\ & x_{i-1} & & & & \\ & & x_i & & & \\ & & & x_{i+1} & & \\ & & & & x_{i+2} & \\ & & & & & \ddots \end{bmatrix}$$

$$[H, X] \neq 0$$

be continued to

$$[H^{(\mathrm{AL})}, X^{(\mathrm{AL})}] = 0$$

$$H_a = \begin{bmatrix} \ddots & & & & & \\ & \varepsilon' & -t' & & & \\ & -t' & \varepsilon' & & & \\ & & & \varepsilon' & -t' & \\ & & & -t' & \varepsilon' & \\ & & & & & \ddots \end{bmatrix}$$

$$X_a = \begin{bmatrix} \ddots & & & & & \\ & x_i' & & & & \\ & & x_i' & & & \\ & & & x_{i+1}' & & \\ & & & & x_{i+1}' & \\ & & & & & \ddots \end{bmatrix}$$

without violating similar restrictions?

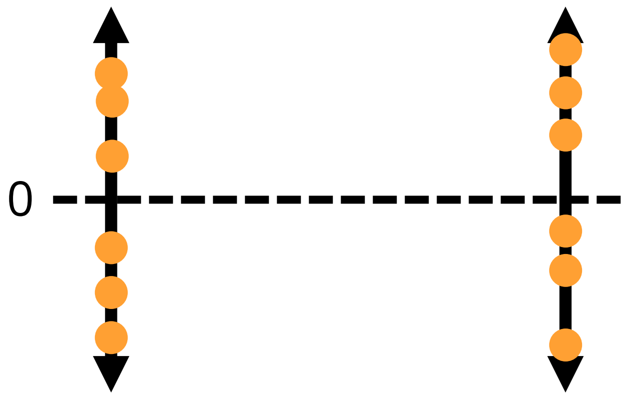
Topology from operators

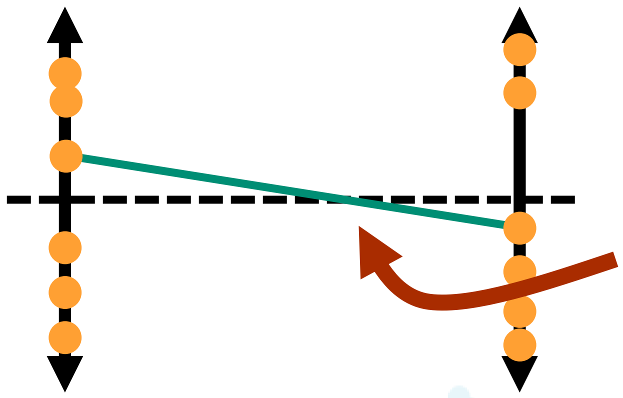
Instead of an invariant, "Can the system be permuted to an *atomic limit*?"

"Can the system's operators be permuted to be *commuting*?"

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices <u>if and only if</u> $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.





Topology from operators

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices <u>if and only if</u> $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.

Theorem (Choi, 1988): If R and S are n-by-n matrices with $RS = SR$, then

$$\mathrm{sig}\begin{bmatrix} R & S \\ S^{\dagger} & -R \end{bmatrix} = 0$$

How do these results help?

- $R \to (H - EI)$
- $S \to \kappa(X - xI) - i\kappa(Y - yI)$

And the requirement that $RS = SR$ becomes

$$[H - EI, X - xI] = 0 \text{ and } [H - EI, Y - yI] = 0$$

Construct the 2D *spectral localizer*:

$$L_{(x,y,E)}(X, Y, H) = \begin{bmatrix} H - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - EI) \end{bmatrix}$$

If $\mathrm{sig}(L_{(x,y,E)}(X, Y, H)) = 0$ for a given E, x, y, then
the system can be continued to the atomic limit at that point.

Topology from operators

Instead of an invariant, "Can the system be permuted to an *atomic limit*?"

"Can the system's operators be permuted to be *commuting*?"

This specific form of the spectral localizer can be generalized

$$L_{(x,y,E)}(X,Y,H) = \begin{bmatrix} H - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - EI) \end{bmatrix}$$

Pauli spin matrices, $\sigma_{x,y,z}$

$$= \kappa(X - xI) \otimes \sigma_x + \kappa(Y - yI) \otimes \sigma_y + (H - EI) \otimes \sigma_z$$

General form uses a non-trivial Clifford representation with sufficient dimensionality

$$L_{(x_1,\ldots,x_d,E)}(X_1, \ldots, X_d, H) = \sum_{j=1}^{d} \kappa(X_j - x_j I) \otimes \Gamma_j + (H - EI) \otimes \Gamma_{d+1}$$

Loring, *Ann. Phys.* **356**, 383 (2015)
Loring and Schulz-Baldes, *New York J. Math.* **23**, 1111 (2017)
Loring and Schulz-Baldes, *J. Noncommut. Geom.* **14**, 1 (2020)

Invariants for other discrete symmetry classes

General form uses a non-trivial Clifford representation with sufficient dimensionality

$$L_{(x_1,\ldots,x_d,E)}(X_1,\ldots,X_d,H) = \sum_{j=1}^{d} \kappa(X_j - x_j I)\otimes\Gamma_j + (H - EI)\otimes\Gamma_{d+1}$$

Different topological invariants are given by this composite operator's properties

Local 1st Chern number $C^L_{(x,y,E)} = \frac{1}{2}\operatorname{sig}\left[L_{(x,y,E)}(X,Y,H)\right] \in \mathbb{Z}$ (2D Class A)

Local 2nd Chern number $C^L_{(x,y,z,w,E)} = \frac{1}{2}\operatorname{sig}\left[L_{(x,y,z,w,E)}(X,Y,Z,W,H)\right] \in \mathbb{Z}$ (4D Class A/AI)

Local winding number $\nu^L_{(x,0)} = \frac{1}{2}\operatorname{sig}\left[\begin{pmatrix} I & 0 \end{pmatrix} L_{(x,0)}(X,H)\begin{pmatrix} 0 \\ \Pi \end{pmatrix}\right] \in \mathbb{Z}$ $\quad H\Pi = -\Pi H$ (1D Class AIII)

Local QSHE number $S^L_{(x,y,E)} = \operatorname{sign}\left[\operatorname{Pf}\left[iQ^\dagger L_{(x,y,E)}(X,Y,H)Q\right]\right] \in \mathbb{Z}_2$ $\quad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I & \sigma_y \\ \sigma_y^\dagger & I \end{pmatrix}$ (2D Class AII)

Loring, *Ann. Phys.* **356**, 383 (2015)
Loring and Schulz-Baldes, *New York J. Math.* **23**, 1111 (2017)
Loring and Schulz-Baldes, *J. Noncommut. Geom.* **14**, 1 (2020)
AC and Loring, *J. Math. Anal. Appl.* **531**, 127892 (2024)

Topological protection from operators

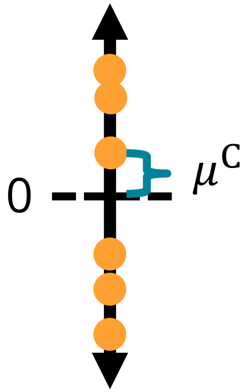
General form uses a non-trivial Clifford representation with sufficient dimensionality

$$L_{(x_1,\ldots,x_d,E)}(X_1,\ldots,X_d,H)=\sum_{j=1}^{d}\kappa\left(X_j-x_jI\right)\otimes\Gamma_j+(H-EI)\otimes\Gamma_{d+1}$$

Also yields a measure of protection (i.e., a "local gap")

$$\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)}=\sigma_{\min}[L_{(x_1,\ldots,x_d,E)}(X_1,\ldots,X_d,H)]$$

(smallest eigenvalue of $L_{(x_1,\ldots,x_d,E)}$)



Spectrum of L

Rigorously,

$$\|\delta H\|<\mu^C$$

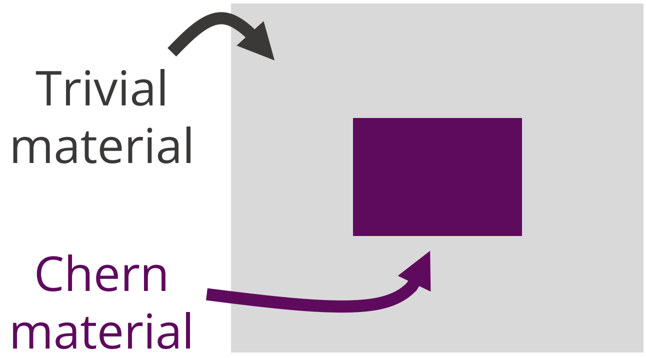
cannot change local topology

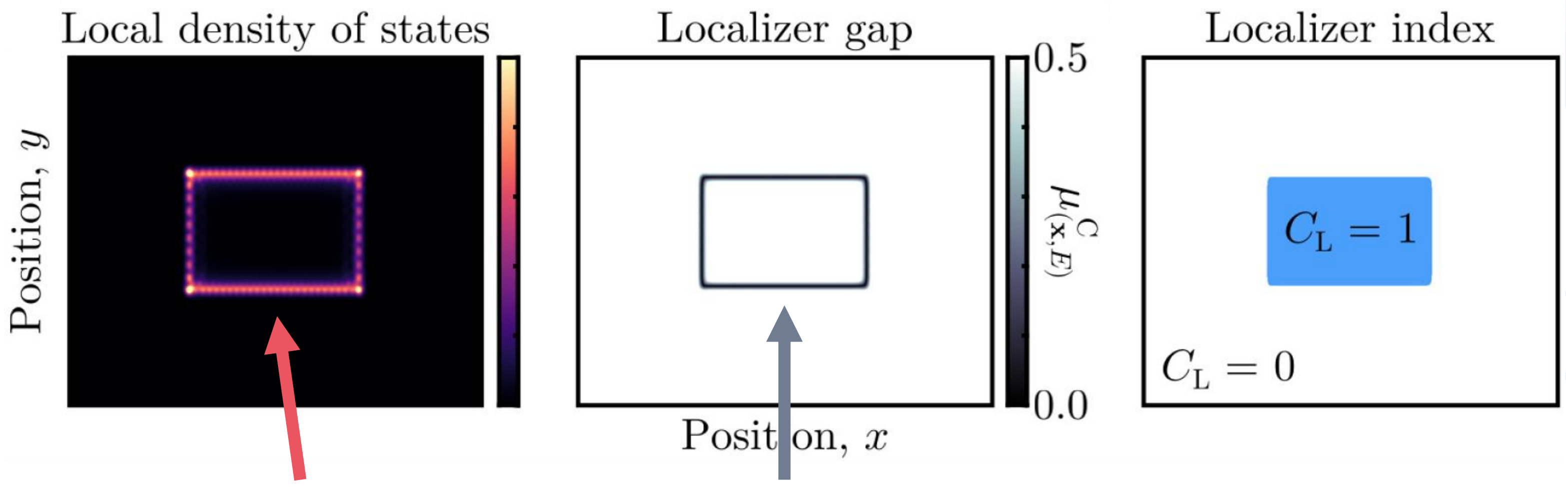
None of the possible local topological markers can change without $\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)}=0$

Loring, *Ann. Phys.* **356**, 383 (2015) – Lemma 7.2
Doll and Schulz-Baldes, *Ann. Phys.* **419**, 168238 (2020)

What does this look like?

Topological heterostructure





Connection between chiral edge states and local gap closing?

- YES!!!
  - Built-in bulk-boundary correspondence
  - Gap closings *necessitate* nearby states of the Hamiltonian

Numerical K-Theory

Have a matrix twice the size of your system

$$L_{(x,y,E)}(X,Y,H) = \begin{bmatrix} H - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - EI) \end{bmatrix}$$

Spectral localizer is sparse

Invariants seem to require knowing its entire spectrum

$$\mathrm{sig}(L_{(x,y,E)}(X,Y,H))$$

How can this possibly be efficient?

Clever matrix factorization.

Fast sparse LDLT publicly available (e.g. MUMPS)

Use LDLT decomposition $L_{(x,y,E)} = NDN^{\dagger}$

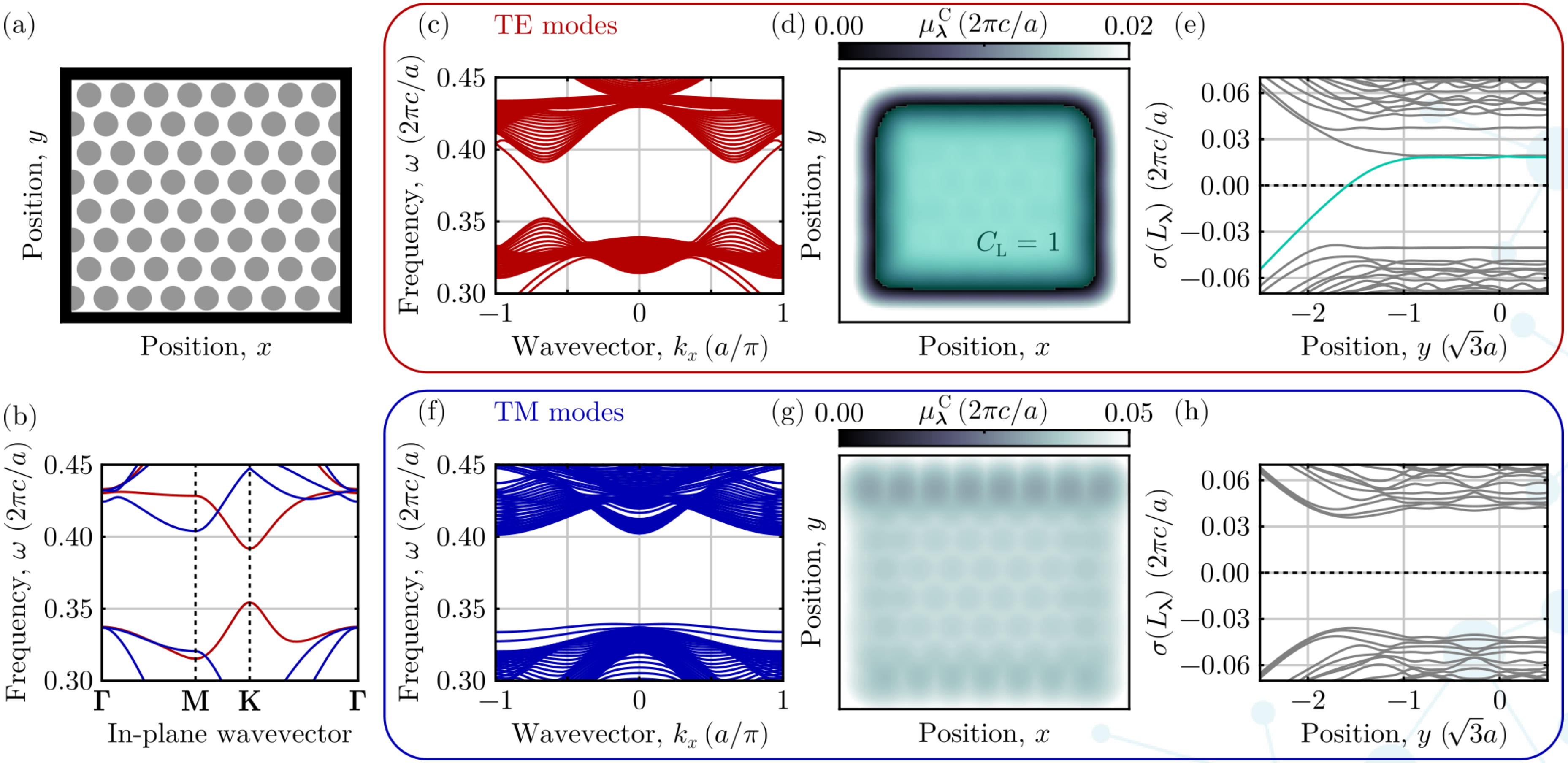
Sylvester's law of inertia $\mathrm{sig}(L_{(x,y,E)}) = \mathrm{sig}(D)$

The Haldane and Raghu photonic Chern insulator

Maxwell's equations

$$L_{(x,y,\omega)}(X,Y,H) = \begin{bmatrix} H - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - \omega I) \end{bmatrix}$$

2D photonic crystal of dielectric pillars in gyro-electric air



Haldane and Raghu, *Phys. Rev. Lett.* **100**, 013904 (2008)
Raghu and Haldane, *Phys. Rev. A* **78**, 033834 (2008)

AC and Loring, *Nanophotonics* **11**, 4765 (2022)

Summary of spectral localizer

- Real-space approach to topology
  - No band structures, Bloch eigenstates, or projections onto an occupied subspace.
- Local markers for any physical dimension, any symmetry class
- Local measure of topological protection

Loring, *Ann. Phys.* **356**, 383 (2015)
Loring and Schulz-Baldes, *New York J. Math.* **23**, 1111 (2017)
Loring and Schulz-Baldes, *J. Noncommut. Geom.* **14**, 1 (2020)

- Built-in bulk-boundary correspondence

Groups are using this already for aperiodic systems

Fulga et al., *Phys. Rev. Lett.* **116**, 257002 (2016)
Franca and Grushin, arXiv:2306.17117

General framework for non-linear topology

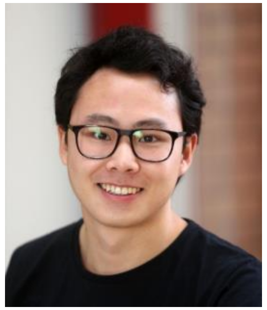
Working in real-space

- Can handle spatial non-linearities for free

$$L_{(x,y,E)}(X,Y,H_{\mathrm{NL}}(\boldsymbol{\psi})) = \begin{bmatrix} H_{\mathrm{NL}}(\boldsymbol{\psi}) - EI & \kappa(X-xI) - i\kappa(Y-yI) \\ \kappa(X-xI) + i\kappa(Y-yI) & -(H_{\mathrm{NL}}(\boldsymbol{\psi}) - EI) \end{bmatrix}$$

On-site non-linearity

$$H_{\mathrm{NL}}(\boldsymbol{\psi}) = H_0 + g|\boldsymbol{\psi}|^2$$



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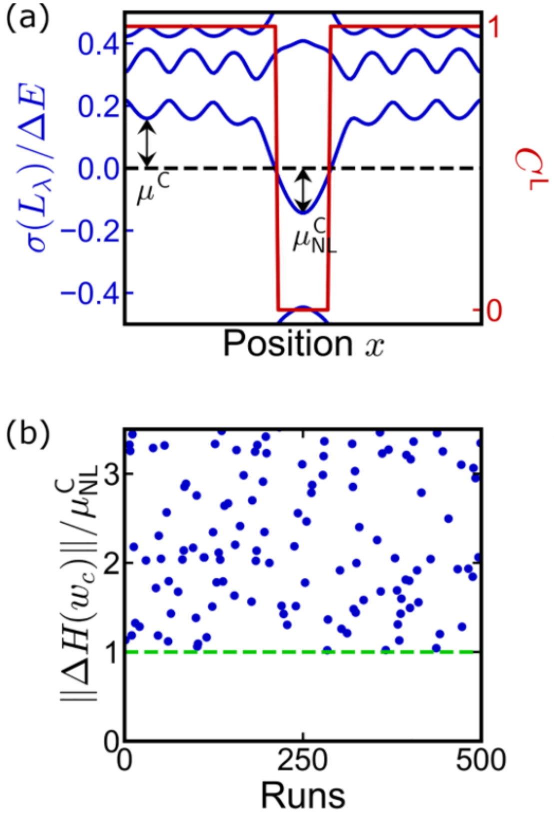
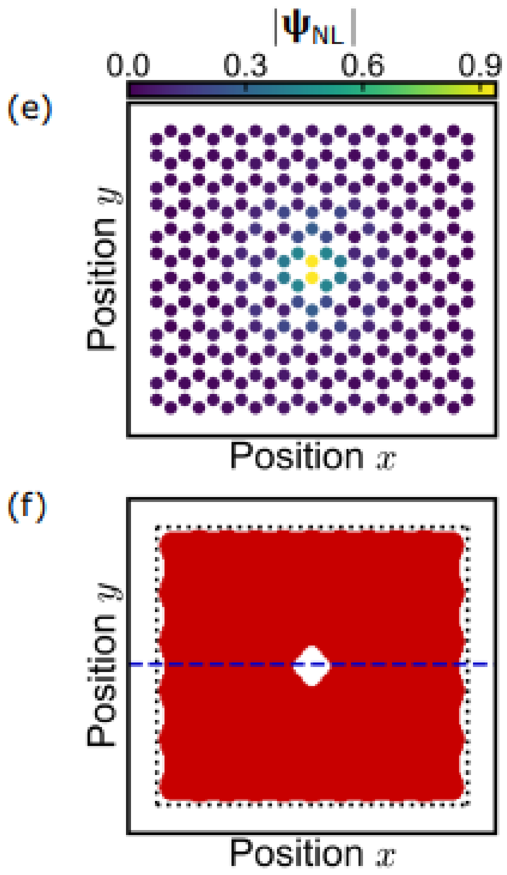
Topological protection now guarantees existence of self-consistent solution



Topological non-trivial nonlinear mode



Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

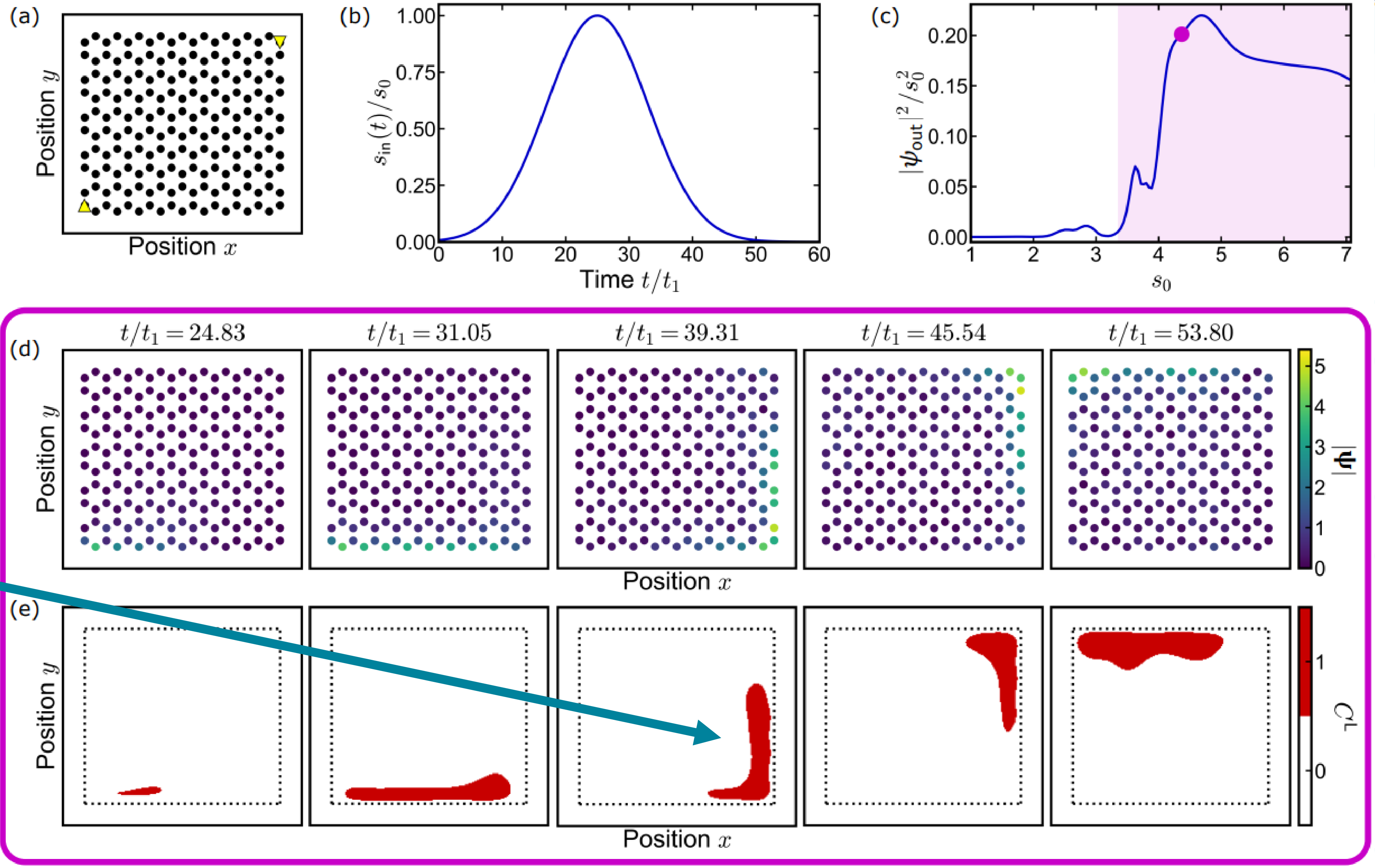
Topological dynamics

Previously predicted and observed edge solitons

Leykam and Chong, *Phys. Rev. Lett.* **117**, 143901 (2016)

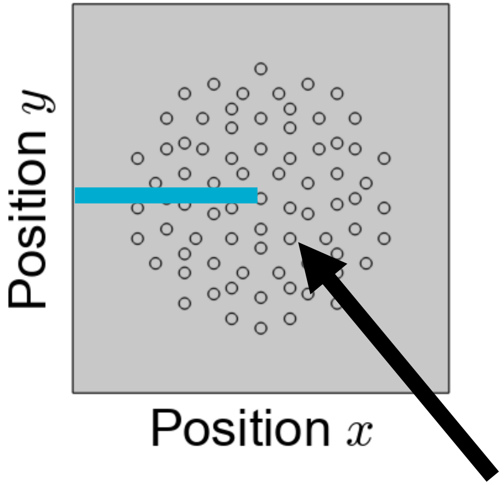
Mukherjee and Rechtsman, *Phys. Rev. X* **11**, 041057 (2021)

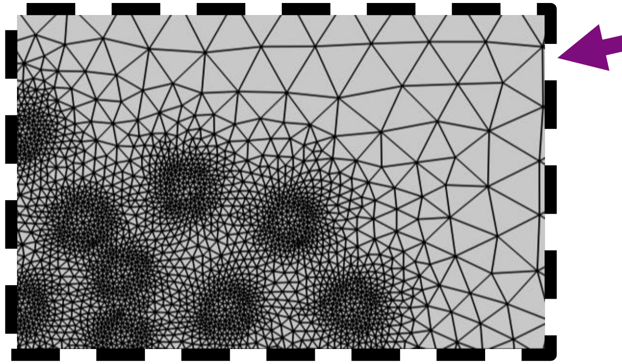
Non-linear topological dynamics!



Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

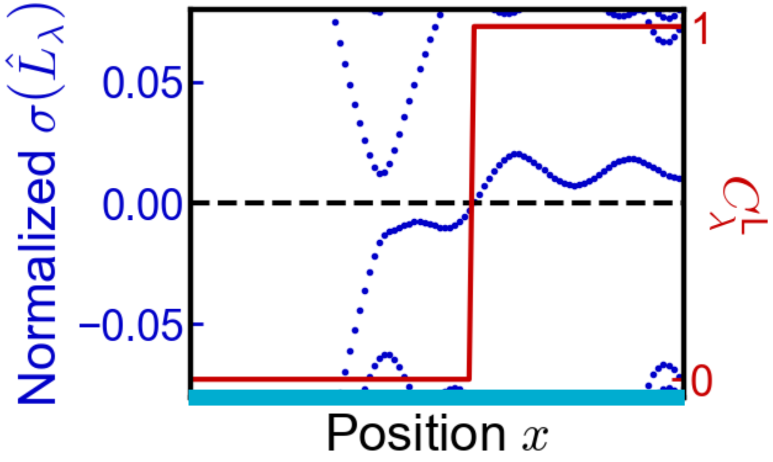
Photonic Chern Quasicrystal



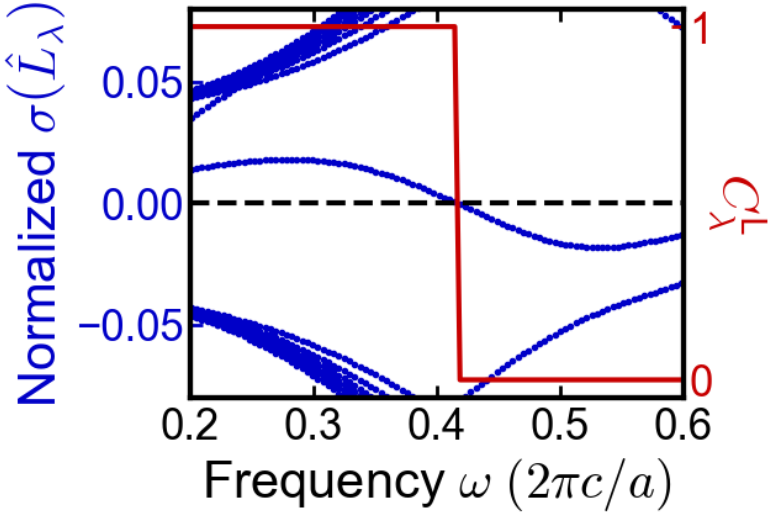


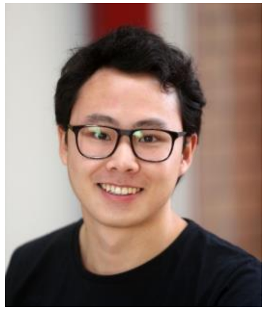
- $L_{\lambda=(x,y,\omega)}(X, Y, H_{\text{eff}})$
- $C_\lambda^L(x,y,\omega) = \frac{1}{2}\text{sig}[L_{(x,y,\omega)}(X,Y,H_{\text{eff}})]$

Vary x, at $\omega = 0.37\ [2\pi c/a]$



Vary ω, at (x_0, y_0)





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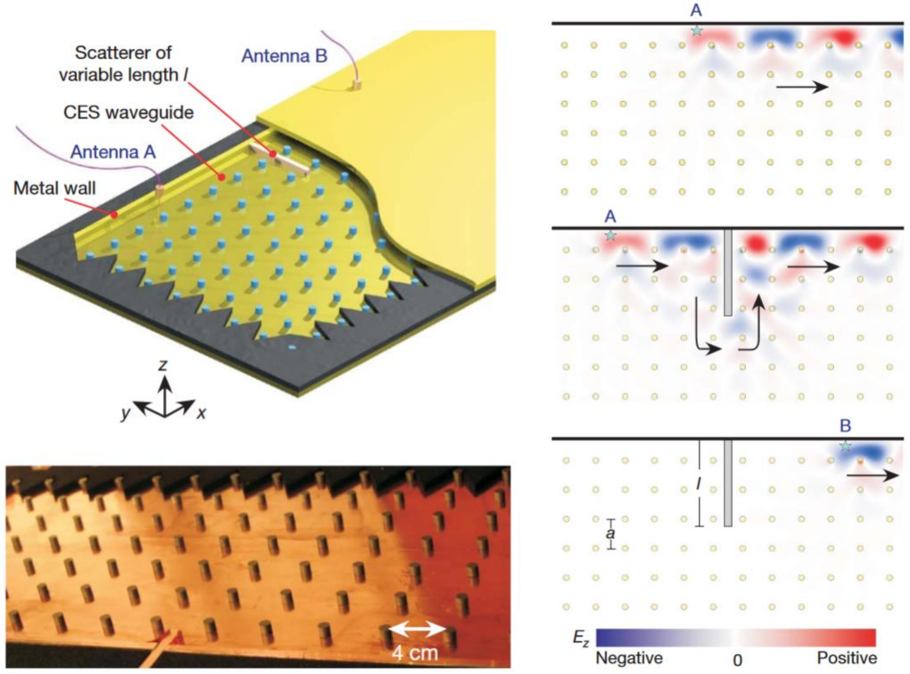
Wong, Loring, and AC, *npj Nanophoton.* **1**, 19 (2024)

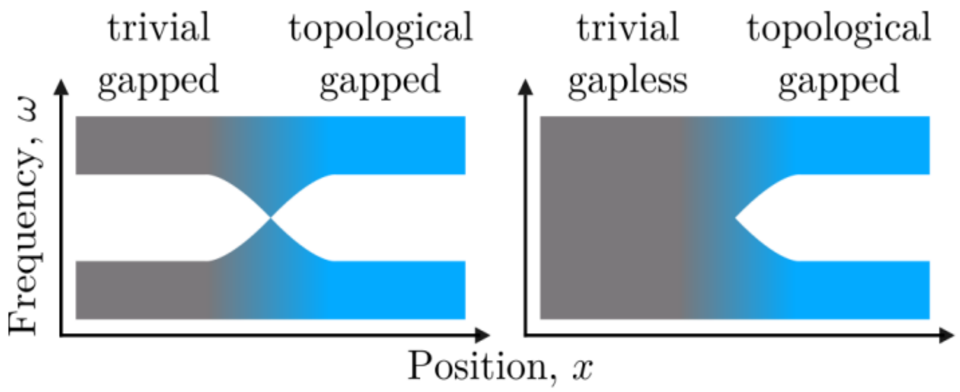
Radiative environments

Realized in microwaves

- Surrounded by a metal
  - Acts as perfect electric conductor



Wang et. al., *Nature* (2009)



Later realizations in other platforms

- Surrounded by air

i.e., radiation

Any topological protection against environment perturbations?

Radiative environments

For Chern number, non-Hermitian generalization is known

$$L_{(x,y,\omega)}(X,Y,H) = \begin{bmatrix} H - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - \omega I)^{\dagger} \end{bmatrix}$$

Yielding

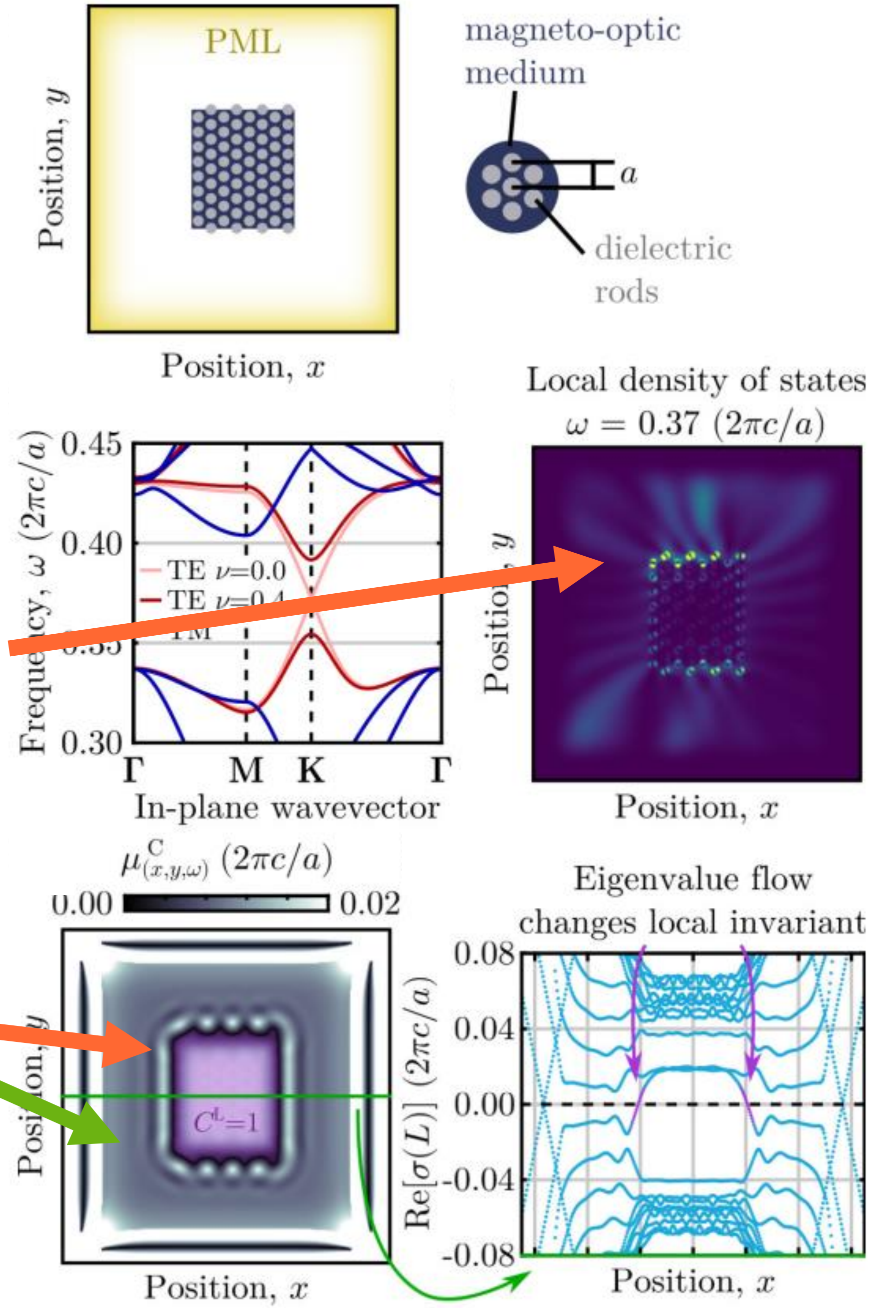
Non-zero local gap!

- Topological protection against perturbations in the environment!

LDOS shows a chiral edge resonance

Spectral localizer proves existence of chiral edge resonance

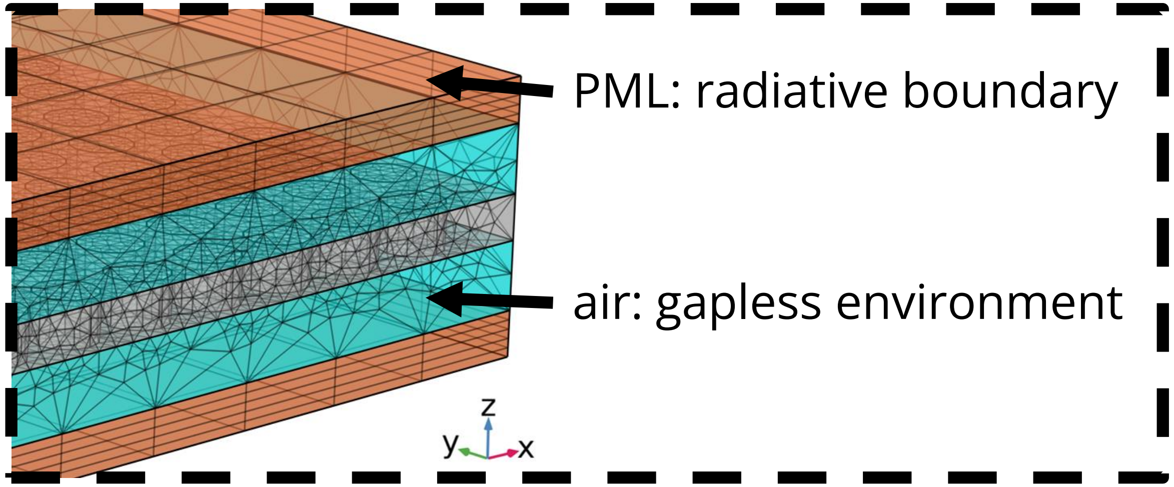
- Resonance… not state.
- Couples to vacuum.



Kahlil Y. Dixon

Dixon, Loring, and AC, *Phys. Rev. Lett.* **131**, 213801 (2023)

Topology in Photonic Crystal Slabs

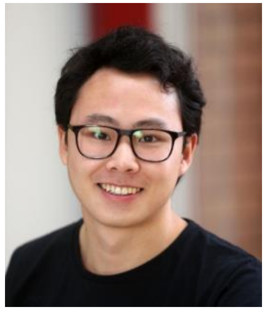


class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	ℤ	0	ℤ	0	ℤ	0	ℤ	0

Schnyder *et. al.*, *Phys. Rev. B* **78**, 195125 (2008)

Topological edge states in slab with 2D strong topological invariant

- Disregard z-direction: $(x, y, z) \rightarrow (x, y)$ (still have all vertices, just "forgetting" about z)
- Look at the change of topology in the (x, y)-plane



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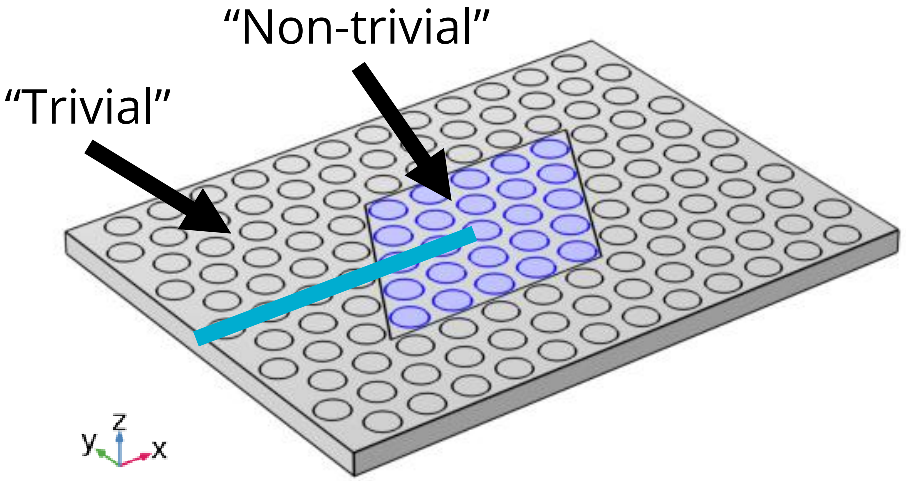
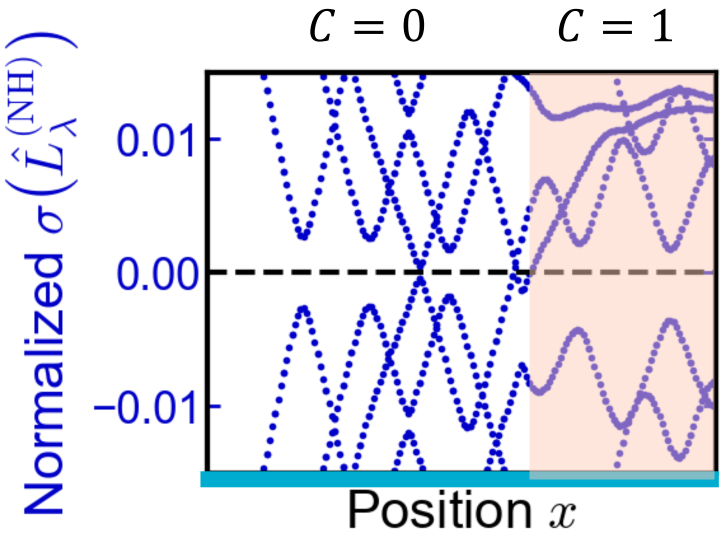




Wong, Loring, and AC, *npj Nanophoton.* **1**, 19 (2024)

Operators don't care about physical meaning

In 1D class AIII (e.g., SSH model), chiral symmetry protects states at $E = 0$

$$H\Pi = -\Pi H, \quad X\Pi = \Pi X, \quad \Pi^2 = I, \quad \Pi = \Pi^\dagger$$

Local winding number:

$$\upsilon_{(x,0)} = \frac{1}{2}\mathrm{sig}[(\kappa(X - xI) + iH)\Pi] \in \mathbb{Z}$$

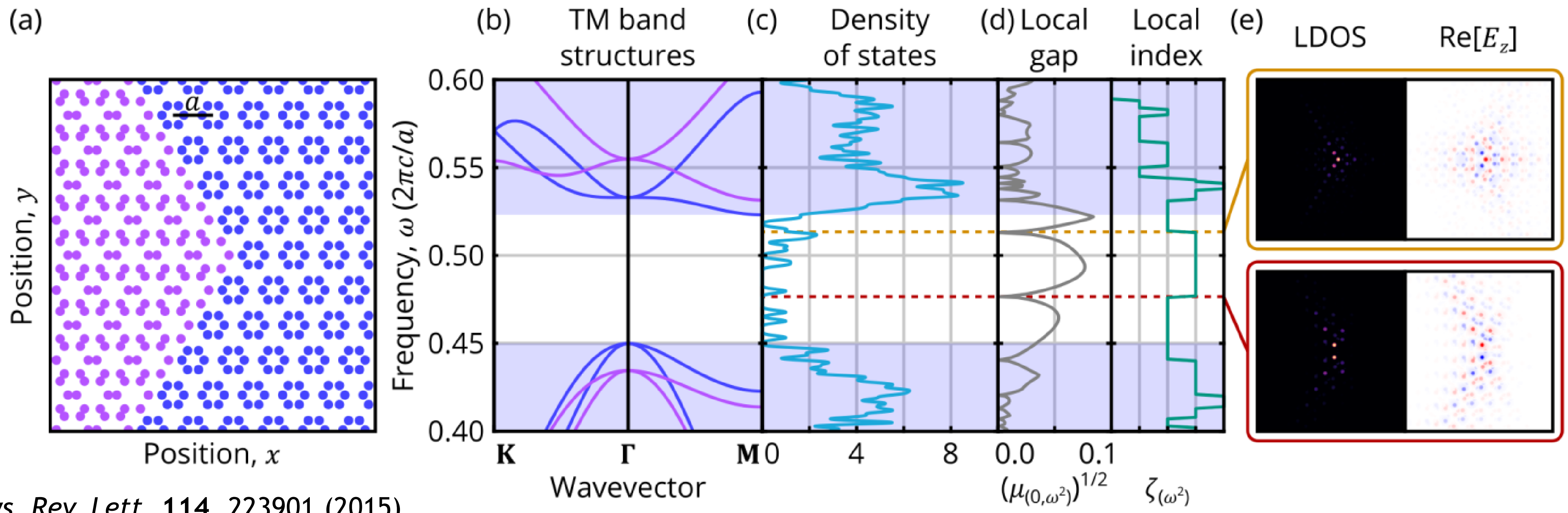
But crystalline symmetry can yield similar commutation relations

$$H\mathcal{S} = \mathcal{S}H, \quad X\mathcal{S} = -\mathcal{S}X, \quad \mathcal{S}^2 = I, \quad \mathcal{S} = \mathcal{S}^\dagger$$

Local "crystalline winding number," protects states at $x = 0$

$$\zeta^{\mathcal{S}}_{(0,E)} = \frac{1}{2}\mathrm{sig}[(H - EI + i\kappa X)\mathcal{S}] \in \mathbb{Z}$$

Local markers for crystalline topology



Wu, Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Smirnova et al., *Phys. Rev. Lett.*, **123**, 103901 (2019)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Local "reflection winding number," protects states at $y = 0$

$$\zeta^{\mathcal{R}_y}_{(0,\omega^2)} = \frac{1}{2}\,\mathrm{sig}\big[(H - \omega I + i\kappa Y)\mathcal{R}_y\big] \in \mathbb{Z}$$

AC, Loring, and Schulz-Baldes, *Phys. Rev. Lett.* **132**, 073803 (2024)

Exciton-Polariton lattices

Direct application to driven-dissipative exciton-polariton systems

$$i\hbar\frac{\partial}{\partial t}\psi = H_0\psi - i\hbar\left(\frac{\gamma_c}{2}\right)\psi + g_c|\psi|^2\psi + \left(g_r + i\hbar\frac{R}{2}\right)n_r\psi + S_{probe}$$
$$\frac{\partial}{\partial t}n_r = -(\gamma_r + R|\psi|^2)n_r + S_{pump}$$

with

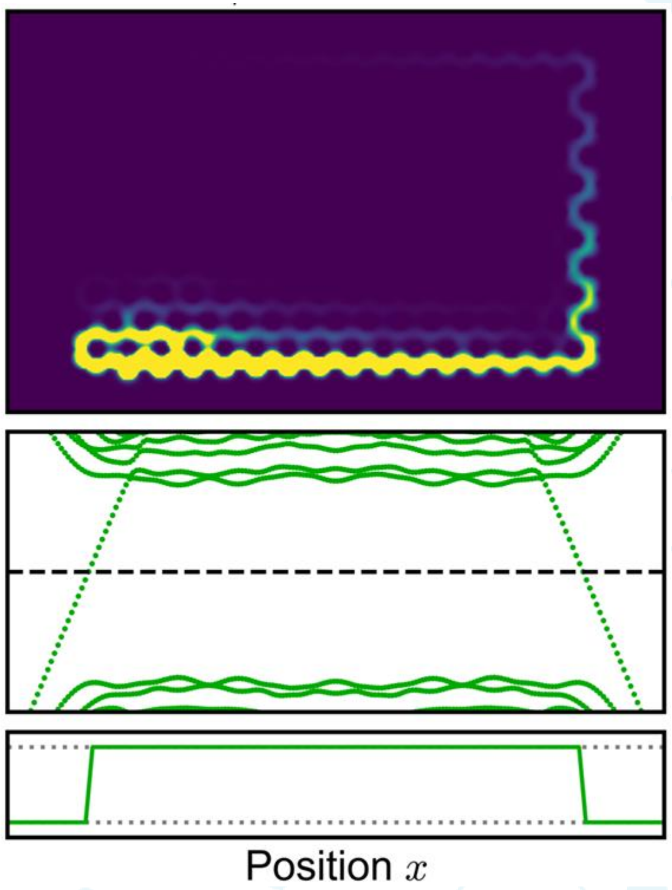
$$H_0 = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + V(x) + \frac{1}{2}\Delta_{eff} & -\beta_{eff}\left(\partial_x - i\partial_y\right)^2 \\ -\beta_{eff}\left(\partial_x + i\partial_y\right)^2 & -\frac{\hbar^2}{2m}\nabla^2 + V(x) - \frac{1}{2}\Delta_{eff} \end{pmatrix}$$

Total Hamiltonian $H(\psi, n_r) = H_0 + i\Gamma(\psi, n_r) + N(\psi, n_r)$

$$L_{(x,y,E)}(X, Y, H) = \begin{bmatrix} H(\psi, n_r) - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H(\psi, n_r) - EI)^\dagger \end{bmatrix}$$

Parameters from

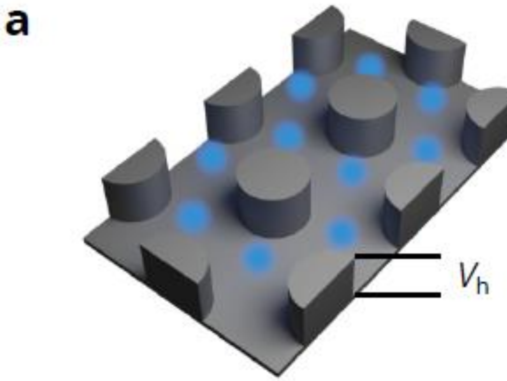
Klembt et al., *Nature* **562**, 552 (2018)

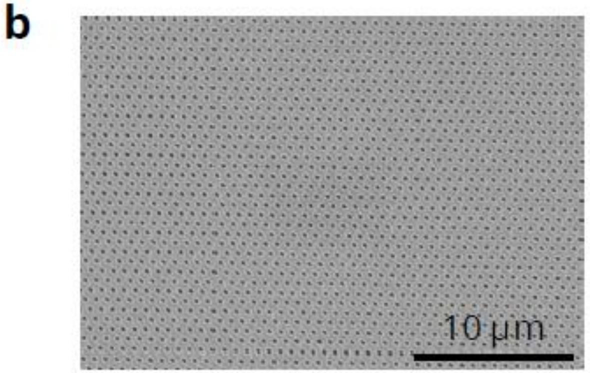


Wong, Betzold, Höfling, AC, in prep.

Application to 2D electron gasses and artificial graphene

Artificial Graphene –
quantum well with added potential V_h



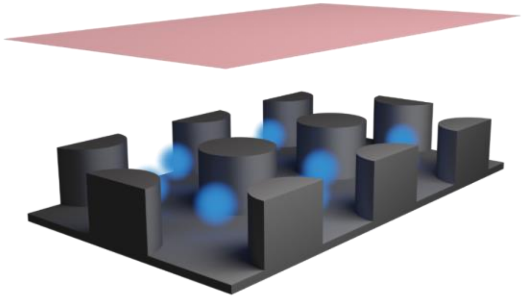


AlSb/InAs/AlSb

$$H = \frac{1}{2m^*}\left(-i\hbar\nabla + e\mathbf{A}(\mathbf{x})\right)^2 + V(\mathbf{x}) - \frac{\mu_B g}{\hbar} s_z B$$

$E_F \approx 4V_h$

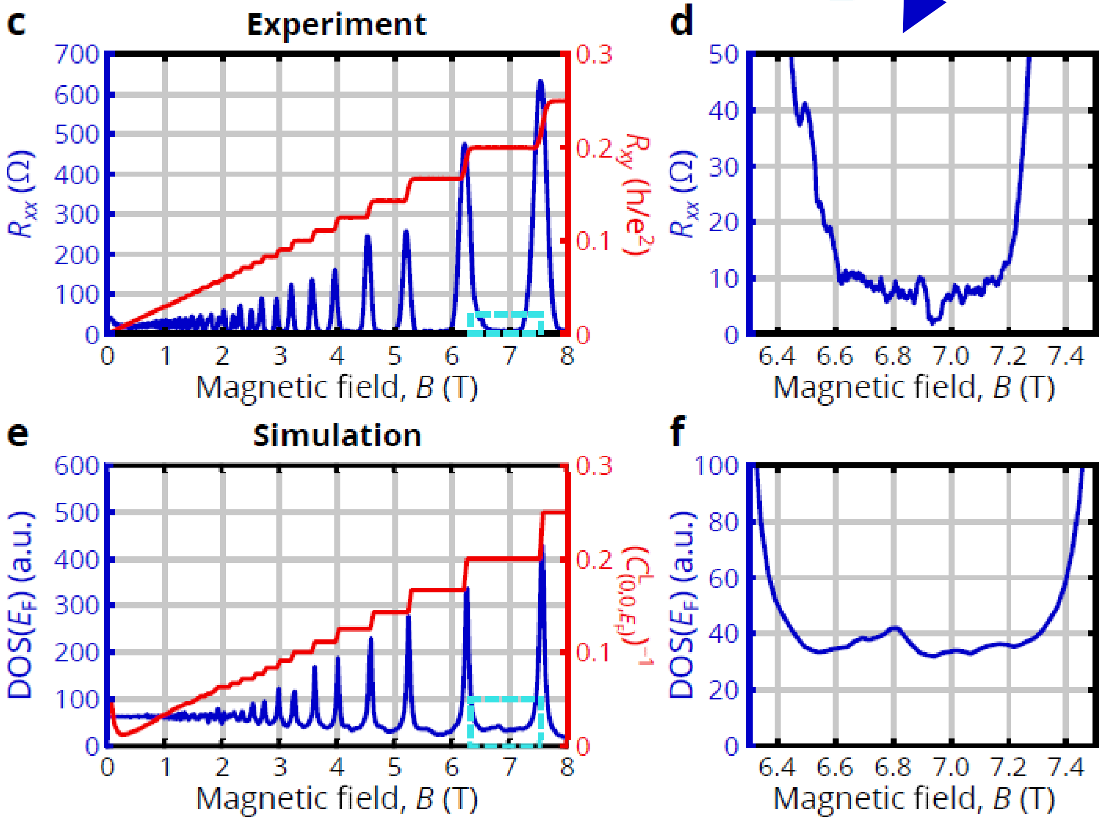
- System mostly behaves as 2D electron gas
  - IQHE



Park et al., *Nano Lett.* **8**, 2920 (2008)
Wunsch et al., *New J. Phys.* **10**, 103027 (2008)

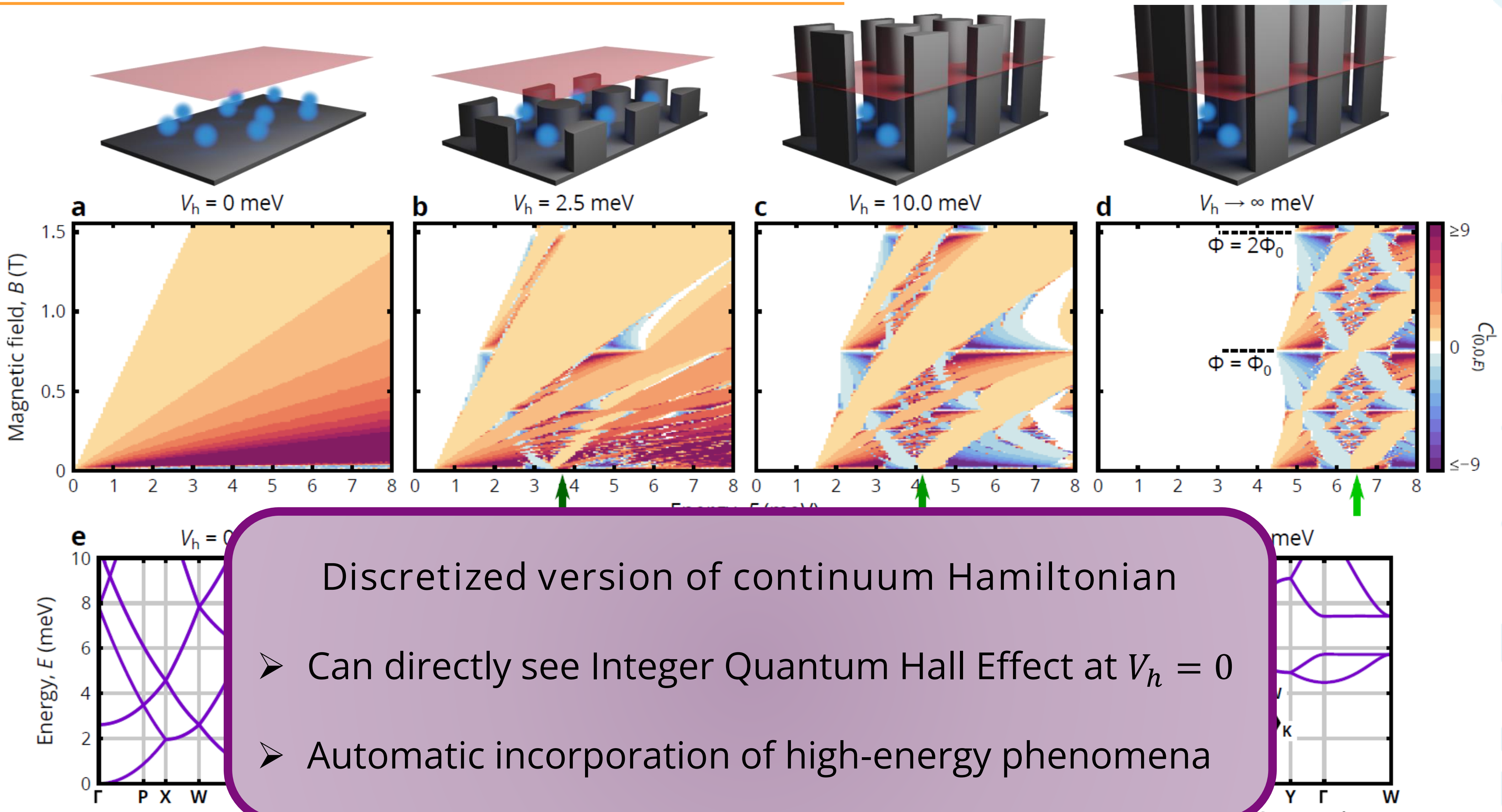
Added potential closes the Landau level gaps

Nevertheless, spectral localizer yields correct Hall resistivity



Spataru, Pan, and AC, in submission

Emergence of Hofstader’s butterfly as potential is turned on



Discretized version of continuum Hamiltonian

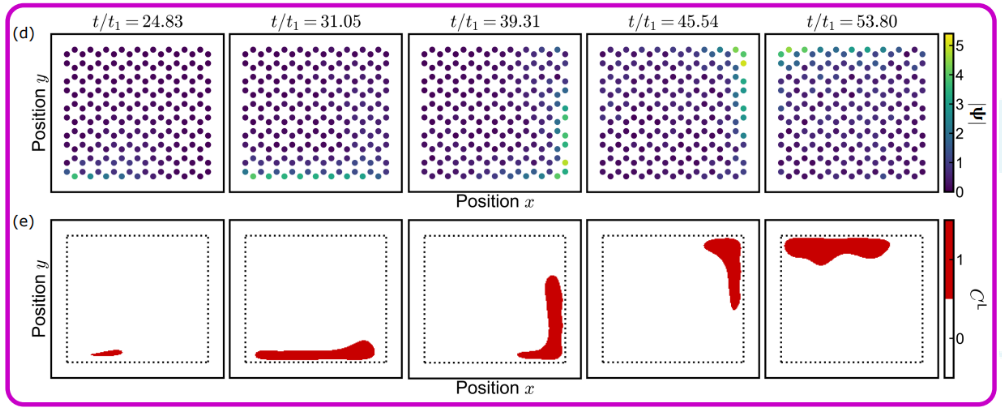
- Can directly see Integer Quantum Hall Effect at $V_h = 0$
- Automatic incorporation of high-energy phenomena

Spataru, Pan, and AC, in submission

Conclusion

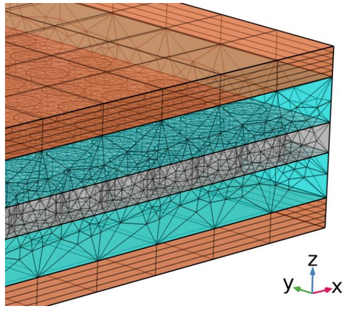
- Topology can be diagnosed using an operator-based framework
  - No band structures or Bloch eigenstates required

$$L_{(x_1,\ldots,x_d,E)} = \sum_{j=1}^{d} \kappa\left(X_j - x_j I\right) \otimes \Gamma_j + (H - EI) \otimes \Gamma_{d+1}$$



- Classifying topological non-linear materials

- Incorporation of radiative boundaries



- Emergence of Hofstadter’s butterfly