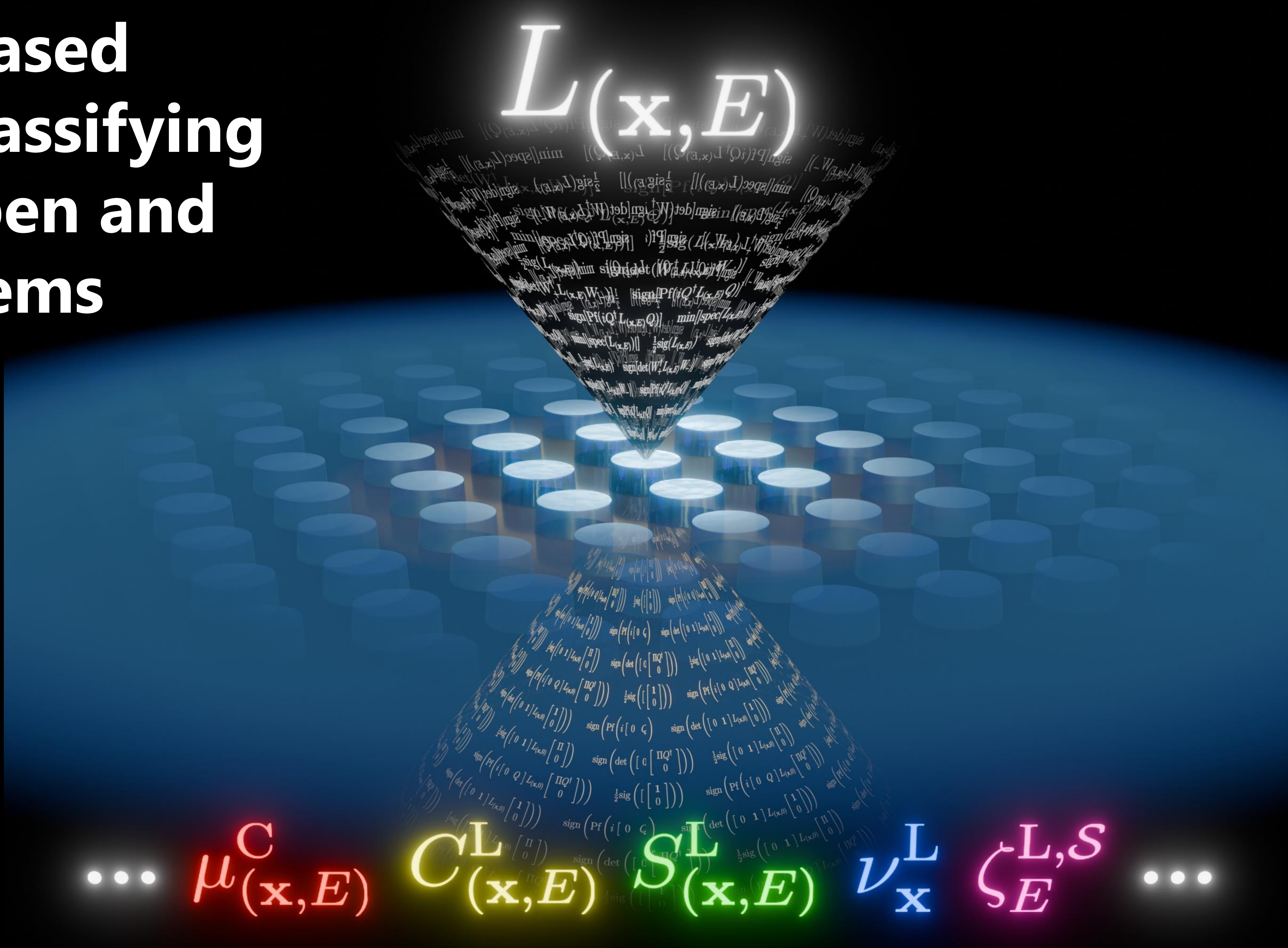
An operator-based approach to classifying topology in open and nonlinear systems



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Topology and Disorder in Wave Physics

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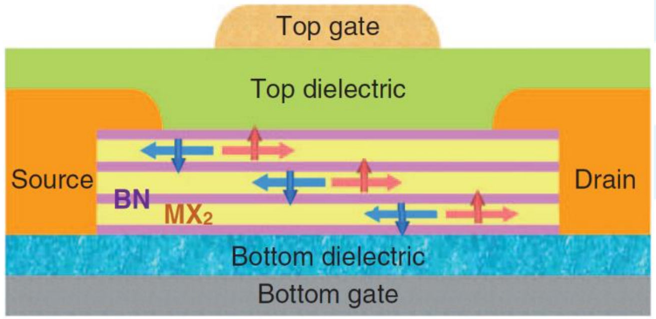
Outline

- An operator-based approach to topological physics
  - Uses a framework called the "***spectral localizer***"
- Topology in gapless systems
  - Emergence of Hofstadter's butterfly
- Classifying topology in non-linear systems
  - Topological dynamics
- Identification of crystalline topology
- Dimensional reduction

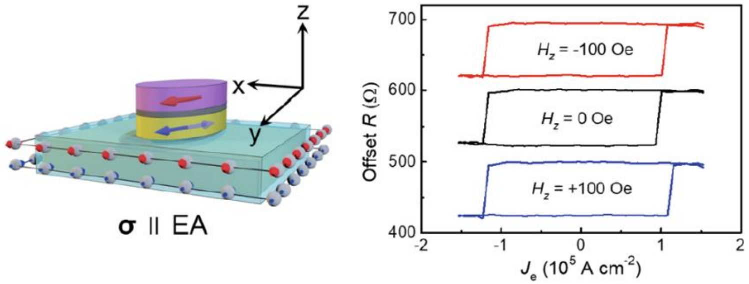
Why are topological materials useful?

- Topological field-effect transistors (TFETs)
  - Dissipation-less transport $\Longrightarrow$ energy efficient



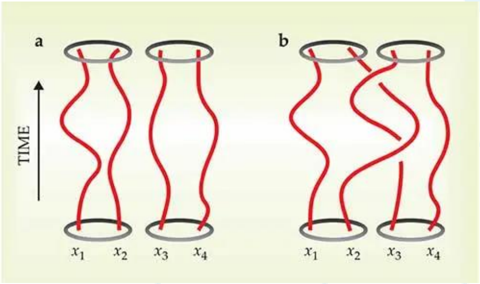
Qian et al., *Science* **346**, 1344 (2014)

- Low-power topological magnetic memory devices
  - Exploit giant spin-orbital torque, reduces power consumption



Wu et al., *Nat. Commun.* **12**, 6251 (2021)

- Fault-tolerant quantum computation
  - Topology greatly reduces need for error correction

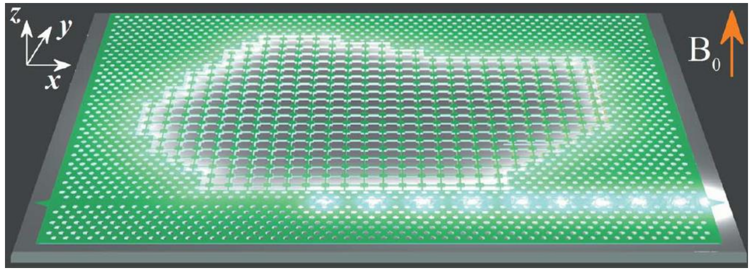


Chen, *APS News* **27**, 4 (2018)

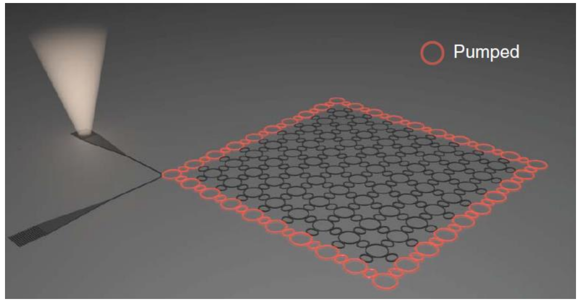
Why make photonics topological?

Topological lasers

- Robust against disorder
- Efficient phase locking

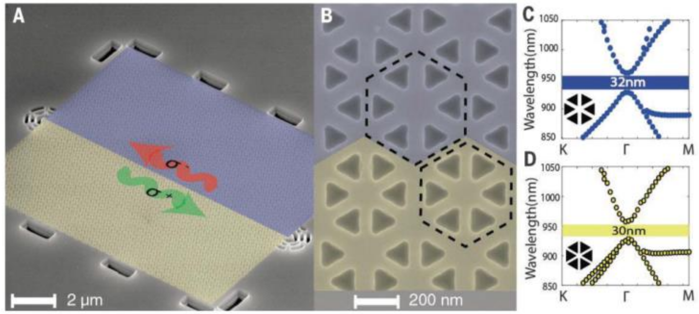


Bahari et al., *Science* **358**, 636 (2017)

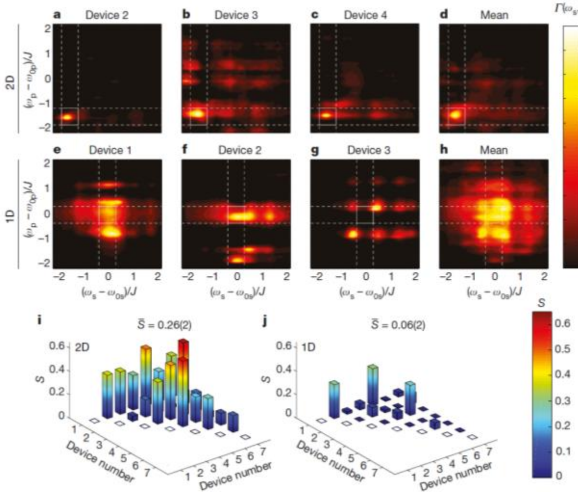


Bandres et al., *Science* **359**, 1231 (2018)

Routing of quantum information

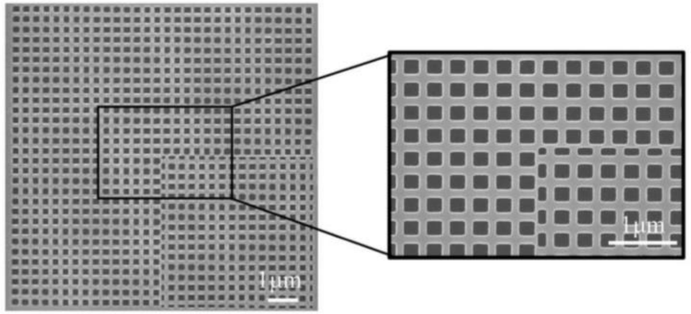


Barik et al., *Science* **359**, 666 (2018)

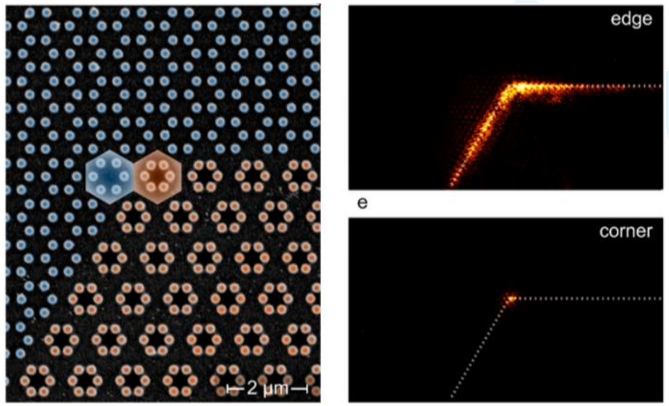


Mittal et al., *Nature* **561**, 502 (2018)

Creating cavities



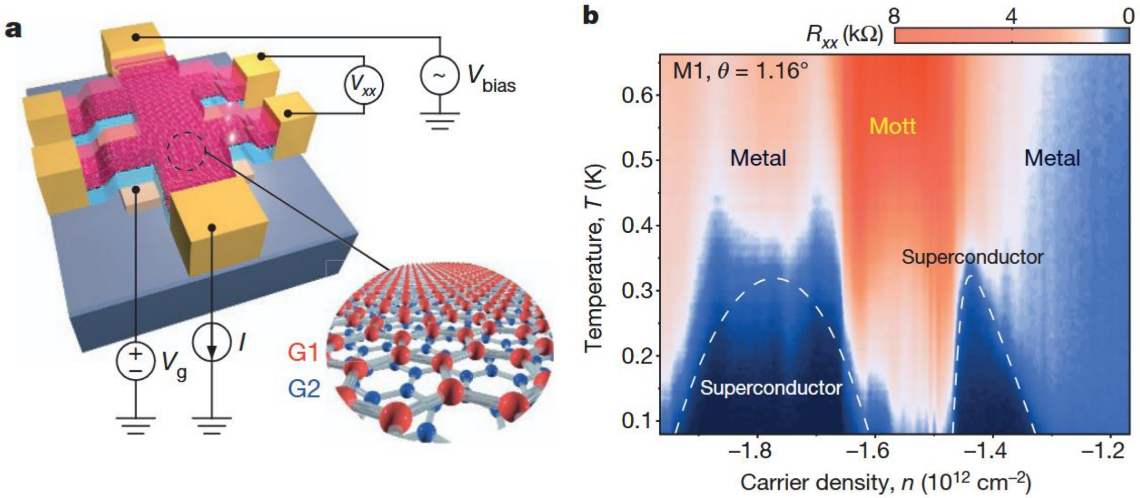
Ota et al., *Optica* **6**, 786 (2019)



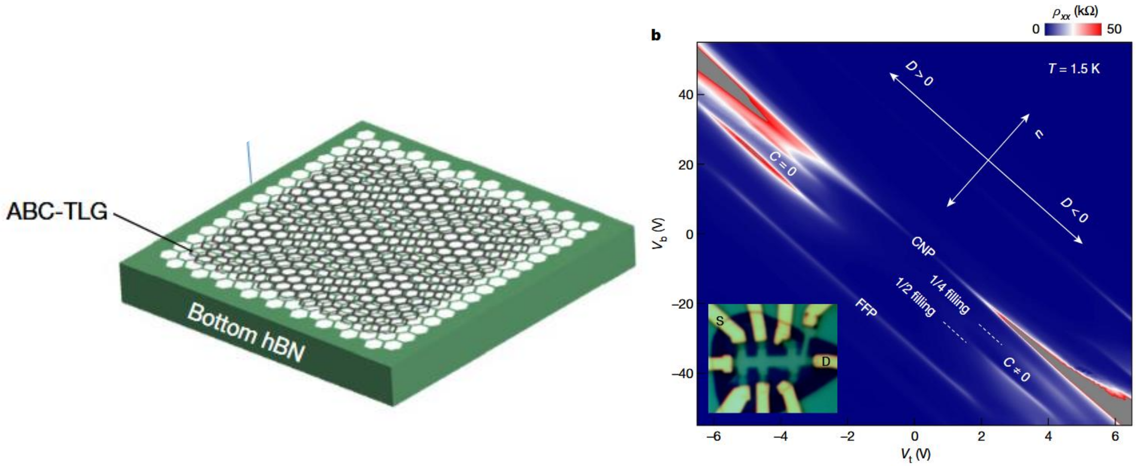
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Topology as a route to delocalized states

Path to correlated phases in electronics

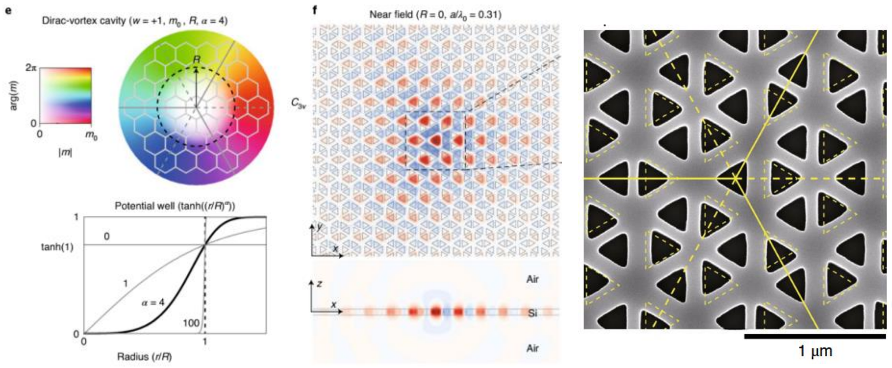


Cao et al., *Nature* **556**, 43 (2018)

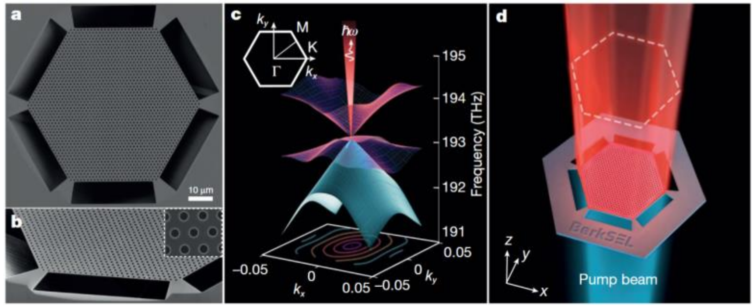


Chen et al., *Nature* **579**, 56 (2020)

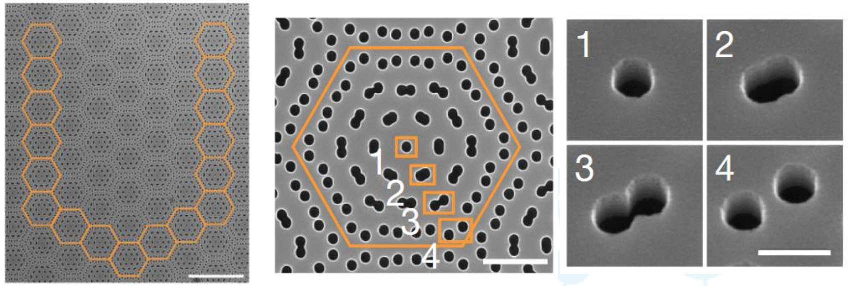
Path to large area lasing in photonics



Gao et al., *Nat. Nanotechnol.* **15**, 1012 (2020)



Contractor et al., *Nature* **608**, 692 (2022)



Luan et al., *Nature* **624**, 282 (2023)

Challenges with invariants

Where band theory can be applied, band theory is awesome.

Where is band theory not applicable (or not useful)?

1) **Material lacks translational symmetry**
   - Quasicrystals
   - Amorphous materials
   - Disorder
   - Finite size effects

2) **Heterostructure lacks a complete or incomplete band gap**
   - Band theory is applicable, but…
     - Not always clear how to calculate the invariant
     - No measure of protection

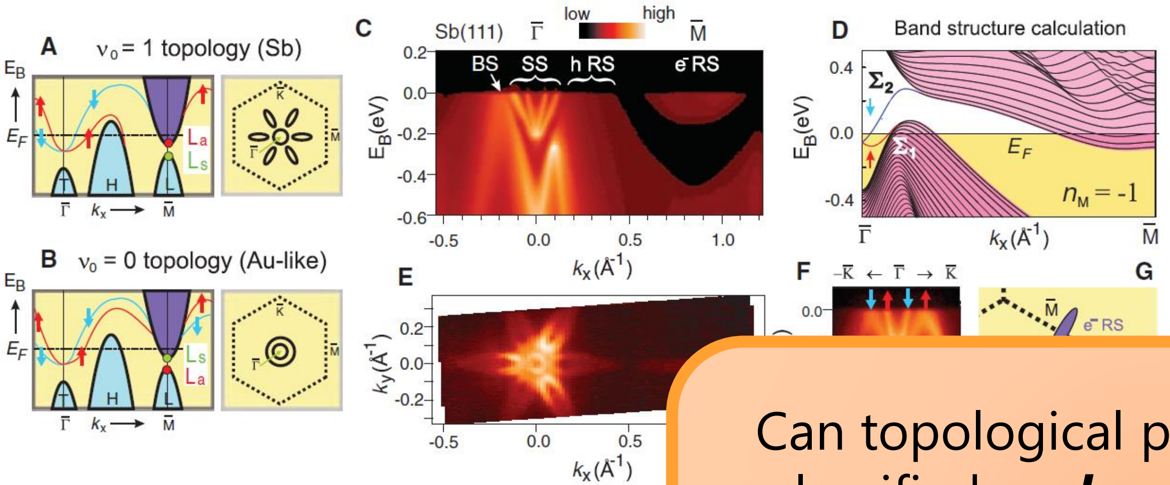
3) **System is non-linear**
   - Localized response breaks translational symmetry

Challenges with invariants in gapless systems

Material may not have a complete bandgap

- ➢ Intervening band may obstruct

Measurements in Sb



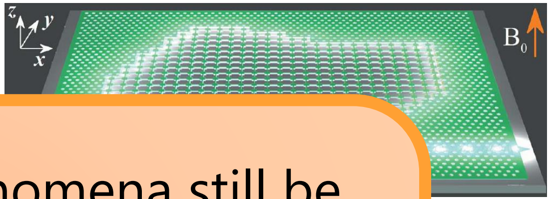
D. Hsieh et al., *Science* 3

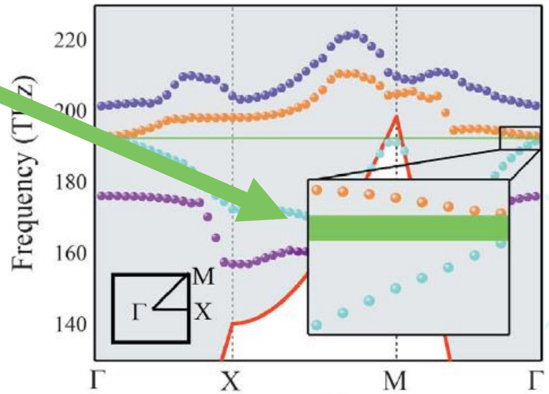
We'd like nanophotonic Chern insulators

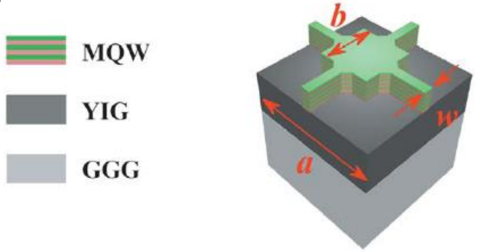
- ➢ Non-reciprocal edge states

But… it's hard to break time-reversal symmetry

Vanishing bandgap (42 pm)







Can topological phenomena still be classified ***and protected*** without a complete band gap?

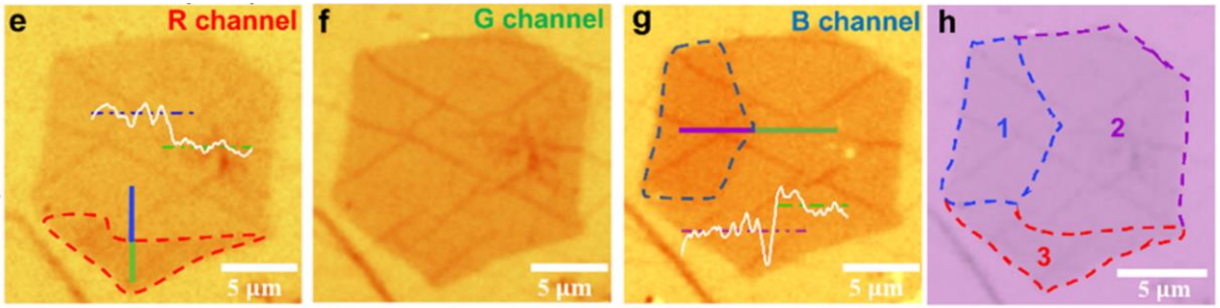
- ➢ Chiral edge resonance?

ahari et al., *Science* **358**, 636 (2017)

Twisted systems can easily become quasicrystals

Twisted materials are never perfect

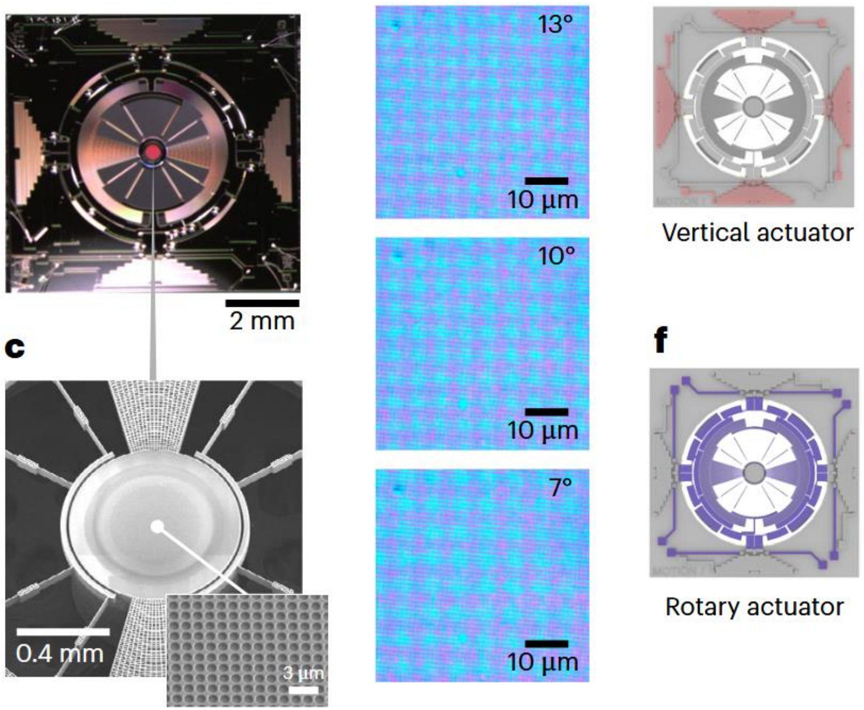
- Wrinkles
- Numerous twist angle domains



Tan et al., *ACS Appl. Mater. Interfaces* **16**, 10867 (2024)

Is there a local measure of topology for all possible rotation angles, disorder?

Arbitrary rotation angles possible



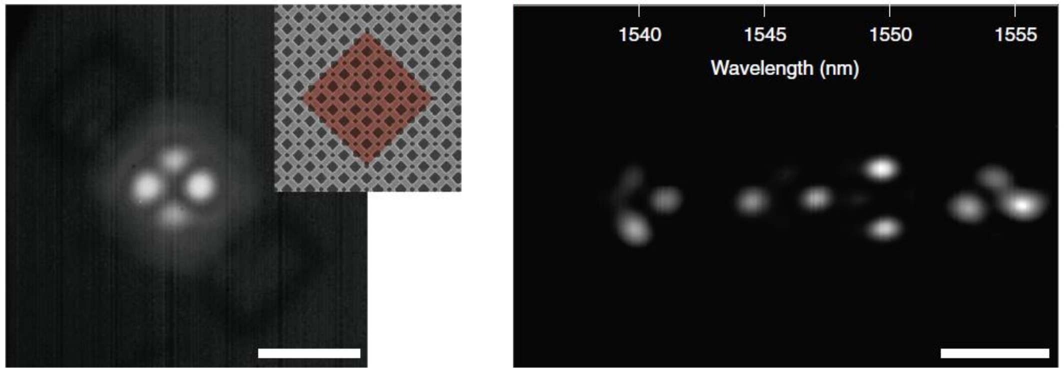
Tang et al., *Nat. Photonics* **19**, 463 (2025)

Lou et al., *Phys. Rev. Lett.* **126**, 136101 (2021)
Tang et al., *Light Sci Appl.* **10**, 157 (2021)
Lou et al., *Sci. Adv.* **8**, add4339 (2022)
Zhang et al., *Nat. Commun.* **14**, 6014 (2023)

Challenges with invariants in finite systems

No current theory for finite systems

How close can two topological cavities be, while maintaining protection?



Kim et al., *Nat. Commun.* **11**, 5758 (2020)

Or how close can two chiral edge states be in a topological Chern system?

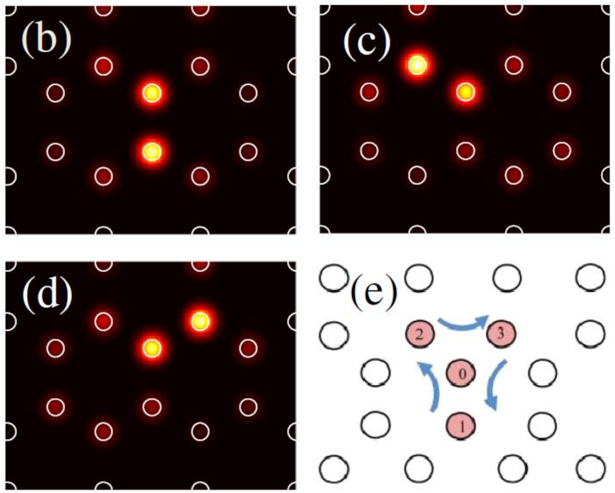
Estimate:

$$e^{-\frac{x}{L}}$$

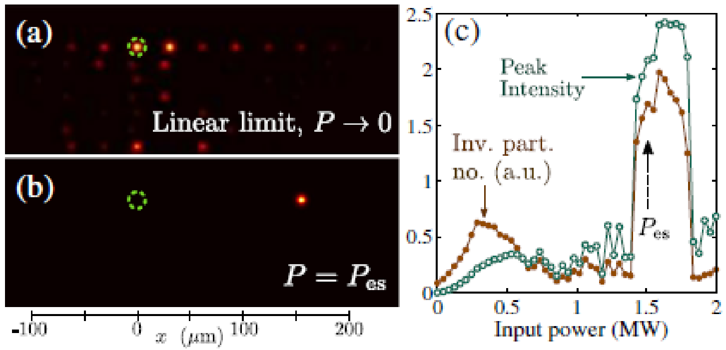
Decay length L set by band gap width ΔE

Is there a local measure of topological protection?

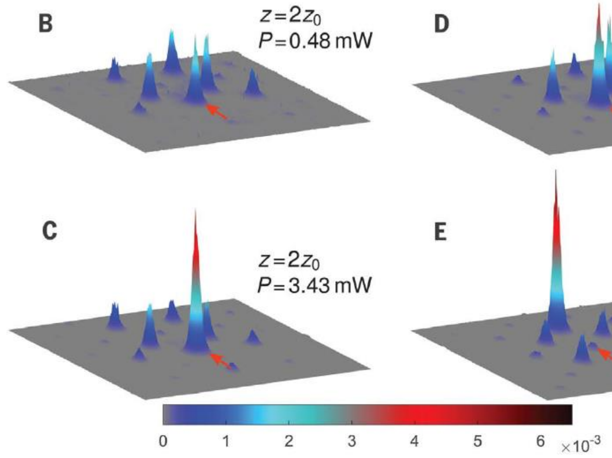
Non-linearities can be local



Lumer et al., *Phys. Rev. Lett.* **111**, 243905 (2013)

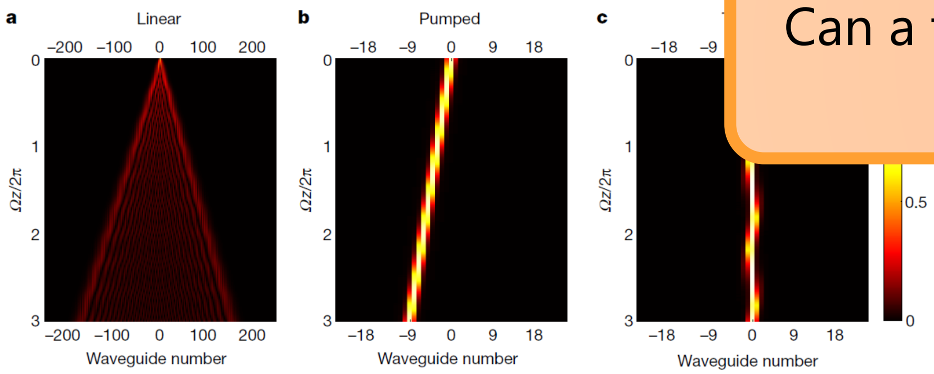


Leykam and Chong, *Phys. Rev. Lett.* **117**, 143901 (2016)

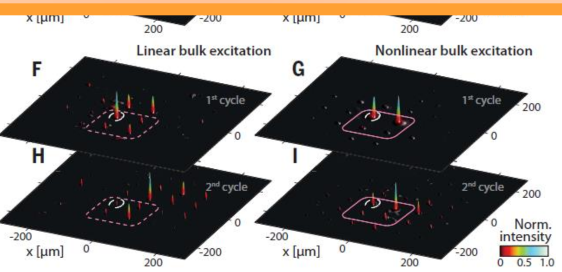


Mukherjee and Rechtsman, *Science* **368**, 856 (2020)

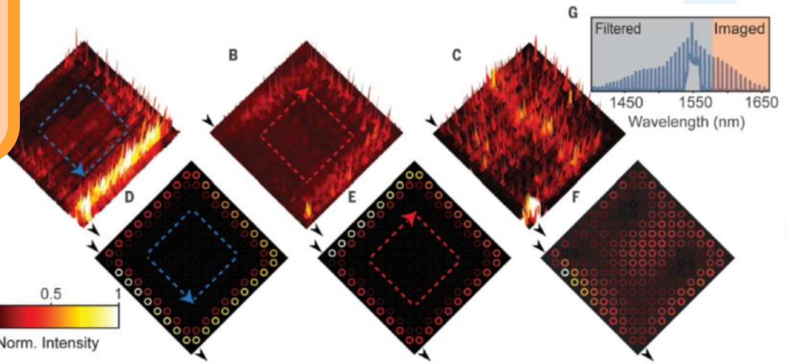
Can a topological invariant be defined without a bulk?



Jürgensen et al., *Nature* **596**, 63 (2021)
Jürgensen et al., *Nat. Phys.* **19**, 420 (2023)

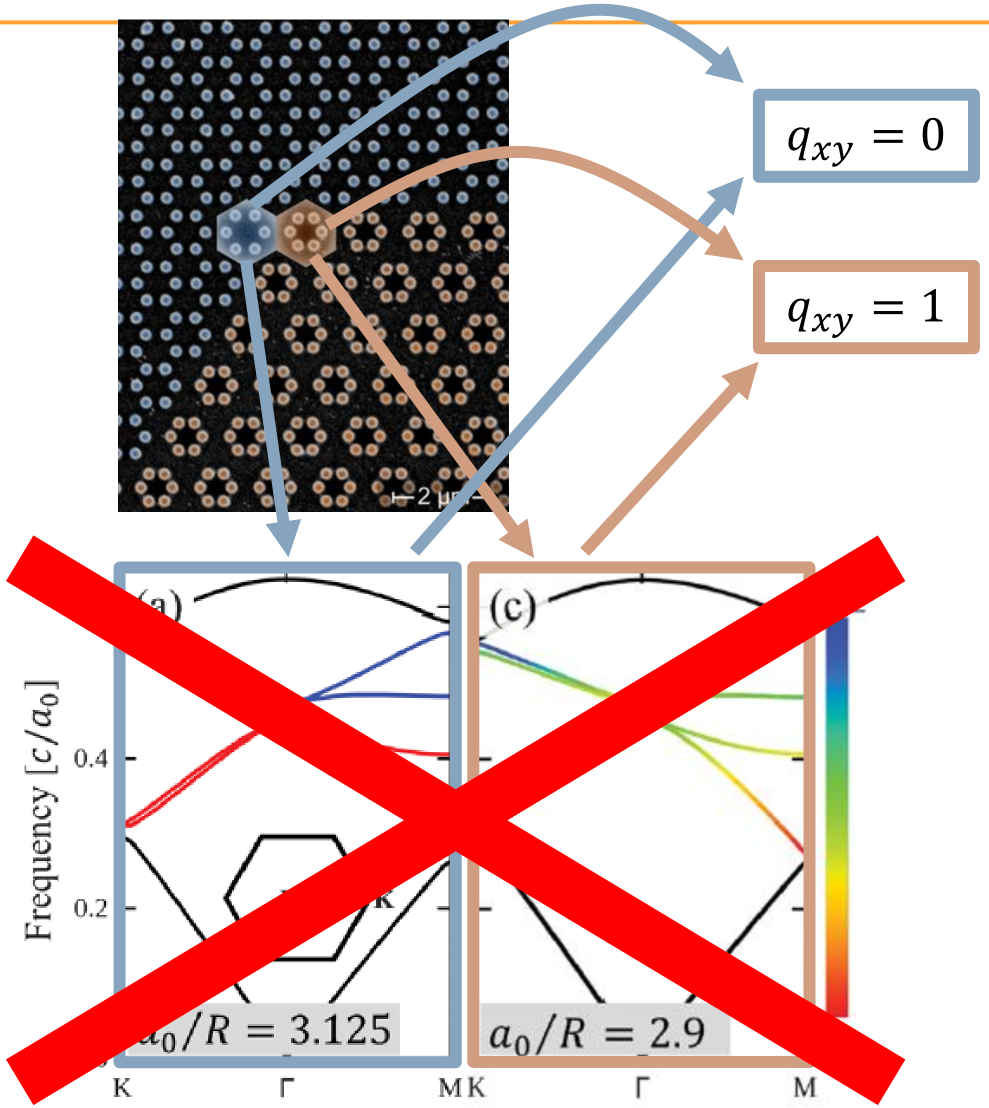


Maczewsky et al., *Science* **370**, 701 (2020)



Flower et al., *Science* **384**, 1356 (2024)

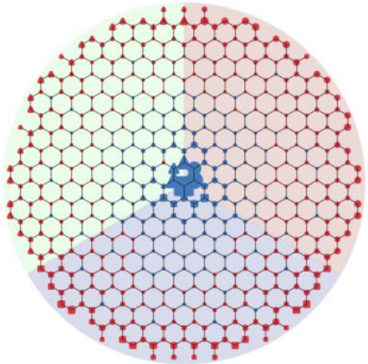
Local real-space approaches to material topology



Wu and Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

Kitaev:

$$\nu(P) = 12\pi i \sum_{j\in A}\sum_{k\in B}\sum_{l\in C}\left(P_{jk}P_{kl}P_{lj} - P_{jl}P_{lk}P_{kj}\right)$$



Kitaev, *Ann. Phys.* **321**, 2 (2006)
Mitchell et al., *Nat. Phys.* **14**, 380 (2018)

Projectors are difficult to calculate for real systems

Neither framework has a local measure of topological protection.

Bianco–Resta:

$$\mathfrak{C}(\mathbf{r}) = -2\pi i \int \left[\tilde{X}(\mathbf{r},\mathbf{r}')\tilde{Y}(\mathbf{r}',\mathbf{r}) - \tilde{Y}(\mathbf{r},\mathbf{r}')\tilde{X}(\mathbf{r}',\mathbf{r})\right] d\mathbf{r}'$$

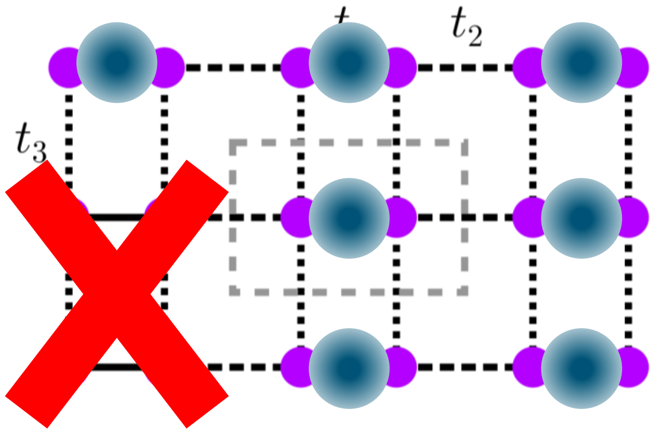
$$\tilde{X}(\mathbf{r},\mathbf{r}') = \int P(\mathbf{r},\mathbf{r}'')x''P(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''$$

Bianco and Resta, *Phys. Rev. B* **84**, 241106(R) (2011)

Implications of topology on the Wannier basis

Systems with non-trivial Chern numbers **DO NOT** possess a complete localized Wannier basis.

$$C_n = \frac{1}{2\pi}\int_{BZ}\left(\frac{\partial A_y^n}{\partial k_x} - \frac{\partial A_x^n}{\partial k_y}\right) d^2\mathbf{k} \neq 0 \qquad \Longleftrightarrow$$



This is an ***if and only if*** statement

- ➢ No complete localized Wannier basis necessitates a non-trivial Chern number.

This generalizes to many other classes of topology

Example: No localized Wannier basis that respects time-reversal symmetry
$\Longleftrightarrow$ non-trivial Kane-Mele invariant (Quantum spin Hall)

Brouder et al., *Phys. Rev. Lett.* **98**, 046402 (2007)

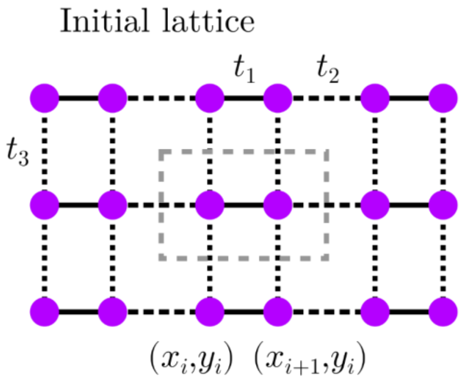
Soluyanov and Vanderbilt, *Phys. Rev. B* **83**, 035108 (2011)

Topology as "Wannierizability"

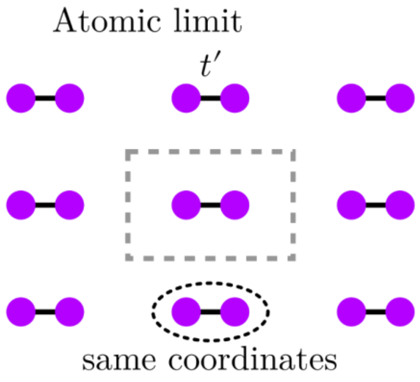
Instead of an invariant, "Does the system possess a complete Wannier basis?"

- Band gap stays open
- Symmetries are preserved

Can a lattice



be continued to



without violating?

If yes
- ➢ Trivial

If no
- ➢ Topological

In other words, "Can the system be permuted to an *atomic limit*?"

(and if multiple inequivalent limits exist, which one?)

➢ Can answer using a lattice's band structure

➢ **Topological quantum chemistry**

Bradlyn et al., *Nature* **547**, 298 (2017)

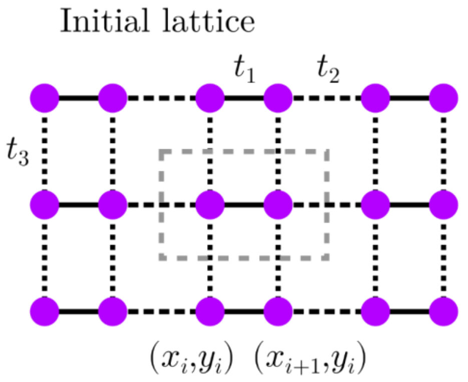
Kitaev, *AIP Conference Proceedings* **1134**, 22 (2009)
Hastings and Loring, *Ann. Phys.* **326** 1699 (2011)
Taherinejad et al., *Phys. Rev. B* **89**, 115102 (2014)
Kruthoff et al., *Phys. Rev. X* **7**, 041069 (2017)
Po et al., *Nat. Commun.* **8**, 50 (2017)

Topology as an atomic limit

Instead of an invariant, "Can the system be permuted to an *atomic limit*?"

- Band gap stays open
- Symmetries are preserved

Can a lattice



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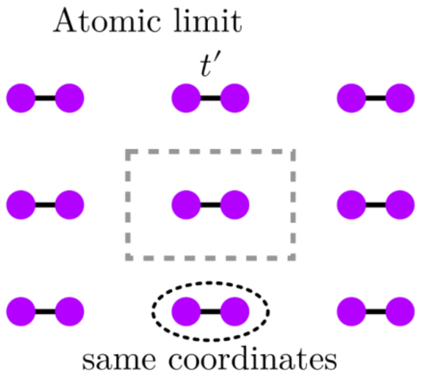


without violating?

If yes

➢ Trivial

If no

➢ Topological

$$H = \begin{bmatrix} \ddots & -t_2 & & -t_3 & & \\ -t_2 & \varepsilon & -t_1 & & -t_3 & \\ & -t_1 & \varepsilon & -t_2 & & -t_3 \\ -t_3 & & -t_2 & \varepsilon & -t_1 & \\ & -t_3 & & -t_1 & \varepsilon & -t_2 \\ & & -t_3 & & -t_2 & \ddots \end{bmatrix} \qquad H_a = \begin{bmatrix} \ddots & & & & & \\ & \varepsilon' & -t' & & & \\ & -t' & \varepsilon' & & & \\ & & & \varepsilon' & -t' & \\ & & & -t' & \varepsilon' & \\ & & & & & \ddots \end{bmatrix}$$

Can the operators

be continued to

without violating similar restrictions?

$$\boxed{[H, X] \neq 0}$$

$$X = \begin{bmatrix} \ddots & & & & & \\ & x_{i-1} & & & & \\ & & x_i & & & \\ & & & x_{i+1} & & \\ & & & & x_{i+2} & \\ & & & & & \ddots \end{bmatrix} \qquad X_a = \begin{bmatrix} \ddots & & & & & \\ & x'_i & & & & \\ & & x'_i & & & \\ & & & x'_{i+1} & & \\ & & & & x'_{i+1} & \\ & & & & & \ddots \end{bmatrix}$$

$$\boxed{\left[H^{(\mathrm{AL})}, X^{(\mathrm{AL})}\right] = 0}$$

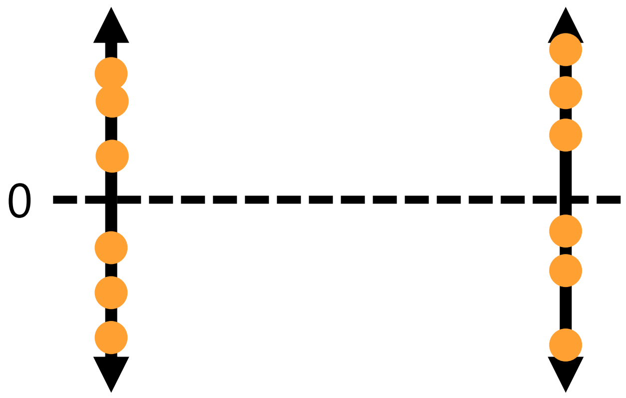
Topology from operators

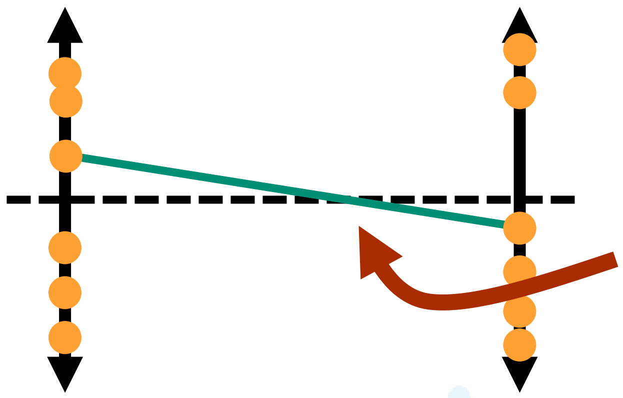
Instead of an invariant, "Can the system be permuted to an *atomic limit*?"

"Can the system's operators be permuted to be *commuting*?"

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices <u>if and only if</u> $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.





Topology from operators

Theorem: Two invertible, Hermitian matrices L and L' can be connected by a path of invertible Hermitian matrices if and only if $\mathrm{sig}(L) = \mathrm{sig}(L')$

- $\mathrm{sig}(L)$ is *signature*, the number of positive eigenvalues minus the number of negative ones.

Theorem (Choi, 1988): If R and S are n-by-n matrices with $RS = SR$, then

$$\mathrm{sig}\begin{bmatrix} R & S \\ S^\dagger & -R \end{bmatrix} = 0$$

How do these results help?

- $R \to (H - EI)$
- $S \to \kappa(X - xI) - i\kappa(Y - yI)$

And the requirement that $RS = SR$ becomes

$$[H - EI, X - xI] = 0 \text{ and } [H - EI, Y - yI] = 0$$

Construct the 2D *spectral localizer*:

$$L_{(x,y,E)}(X, Y, H) = \begin{bmatrix} H - EI & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -(H - EI) \end{bmatrix}$$

If $\mathrm{sig}(L_{(x,y,E)}(X, Y, H)) = 0$ for a given E, x, y, then

the system can be continued to the atomic limit at that point.

Topology from operators

Intuitively… what's going on?

$$L_{(x,y,E)}(X,Y,H) = \kappa(X - xI) \otimes \sigma_x + \kappa(Y - yI) \otimes \sigma_y + (H - EI) \otimes \sigma_z$$

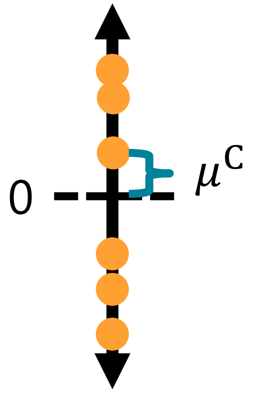
- H and X, Y contain "orthogonal" information
- Pauli matrices (+ identity) form a basis for 2-by-2 Hermitian matrices.

➢ Combination preserves the independent information in H and X, Y while forming a single matrix.

Measure of protection (i.e., a "local gap")

$$\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)} = \sigma_{\min}[L_{(x_1,\ldots,x_d,E)}(X_1,\ldots,X_d,H)]$$

(smallest eigenvalue of $L_{(x_1,\ldots,x_d,E)}$)



Spectrum of L

Rigorously,

$$\|\delta H\| < \mu^C$$

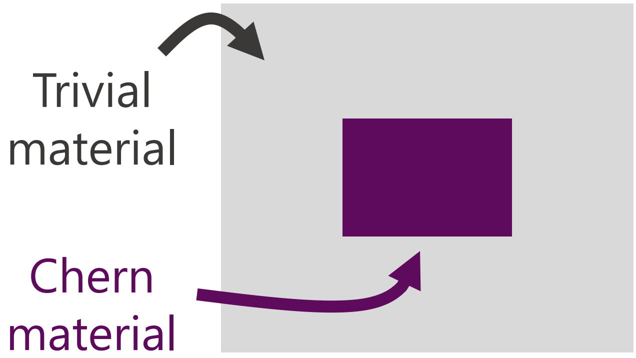
cannot change local topology

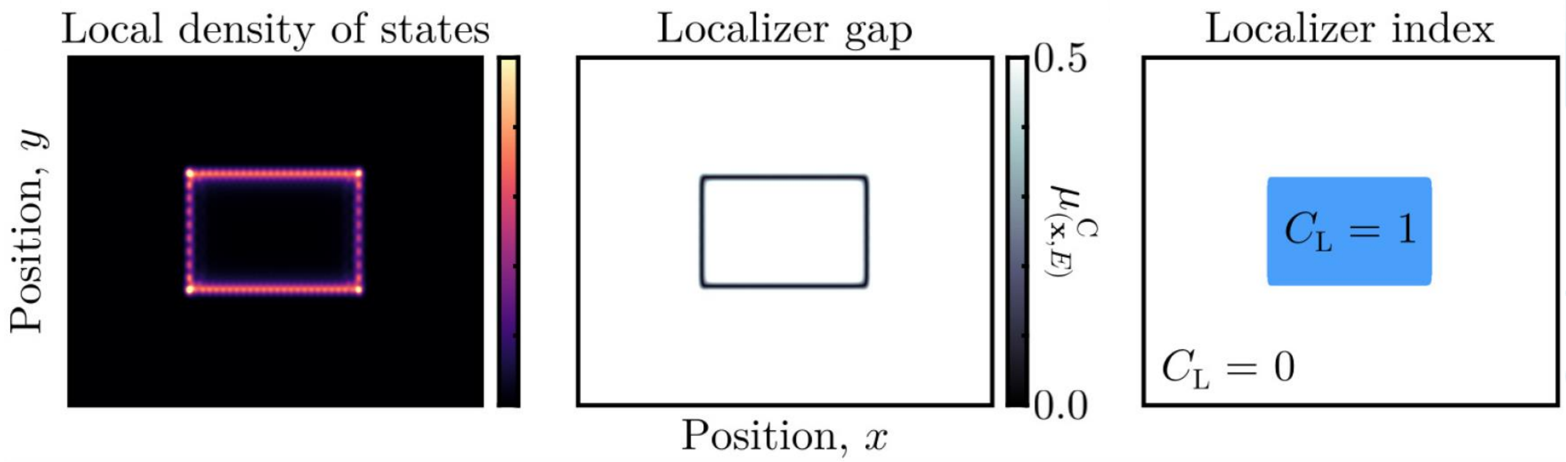
(Weyl's inequality)

Loring, *Ann. Phys.* **356**, 383 (2015)
Loring and Schulz-Baldes, *New York J. Math.* **23**, 1111 (2017)
Loring and Schulz-Baldes, *J. Noncommut. Geom.* **14**, 1 (2020)

What does this look like?

Topological heterostructure





- Built-in bulk-boundary correspondence

  $\mu_{(\boldsymbol{x},E)}(\boldsymbol{X}, E) \approx 0 \;\Rightarrow\;$ there exists some $|\phi\rangle$

  that satisfies $\boldsymbol{X}|\phi\rangle \approx \boldsymbol{x}|\phi\rangle$ and $H|\phi\rangle \approx E|\phi\rangle$ with bounded variances.

- Thus, local gap closings ⇒ nearby states

Implementation is surprisingly fast

E.g., use LDLT decomposition $\longrightarrow L_{(x,y,E)} = NDN^{\dagger}$

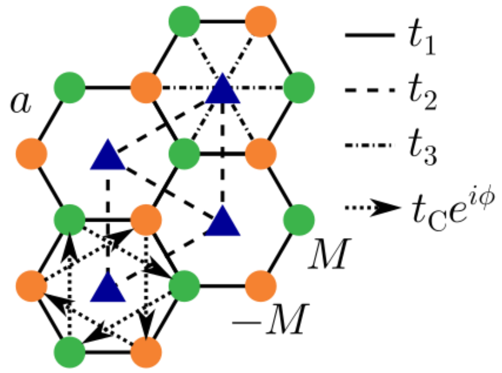
Sylvester's law of inertia $\longrightarrow \mathrm{sig}(L_{(x,y,E)}) = \mathrm{sig}(D)$

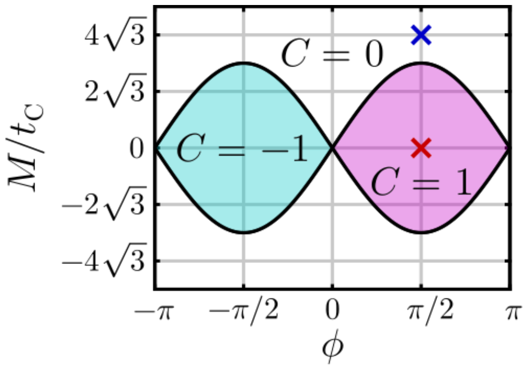
Fast sparse LDLT publicly available (e.g. MUMPS)

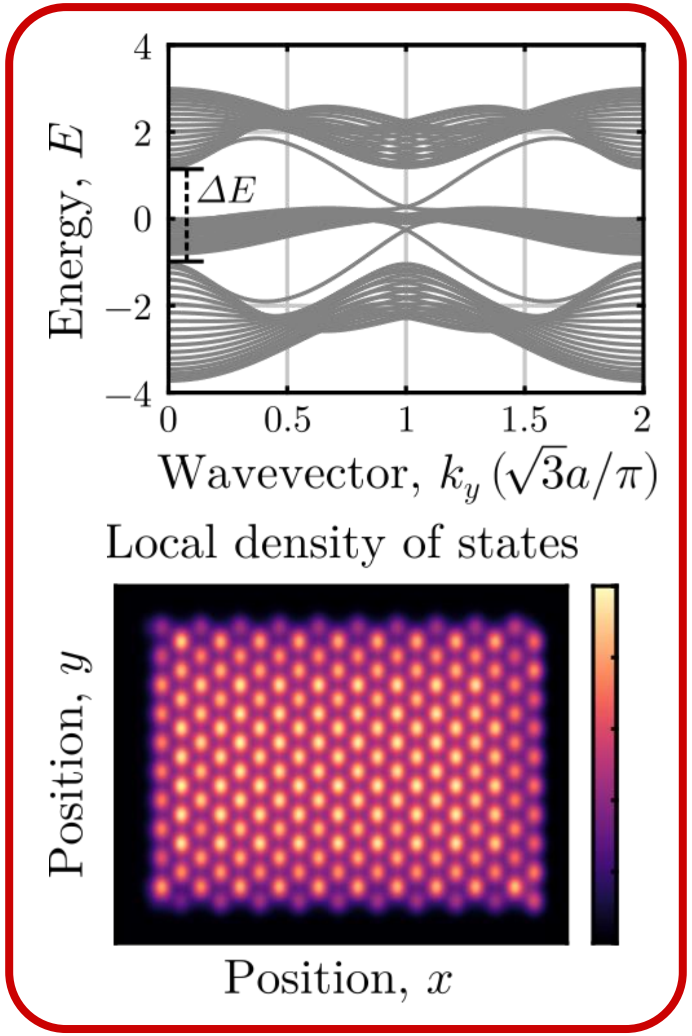
AC, Loring, and Vides, *J. Math. Phys.* **64**, 023501 (2023)

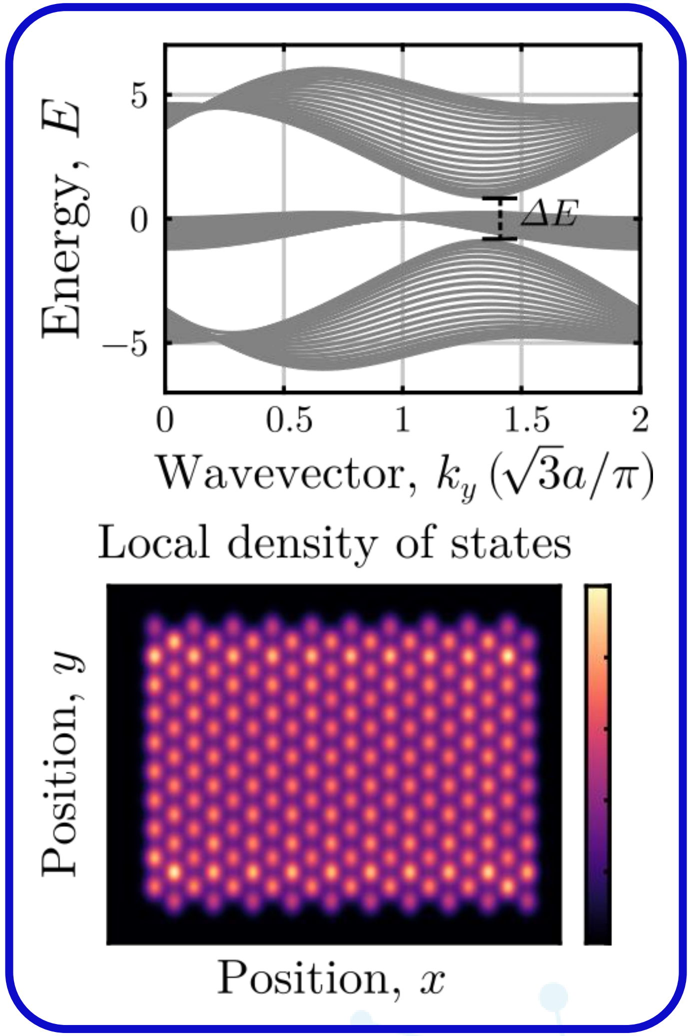
Topology for a gapless system?

“metallized Haldane model”





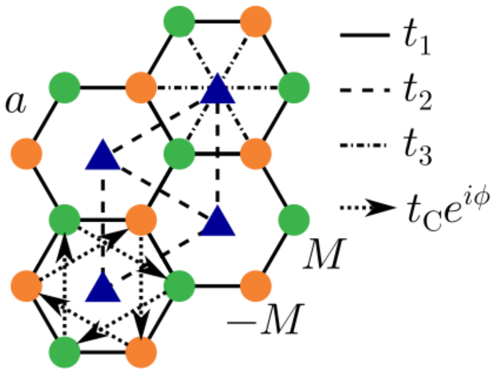


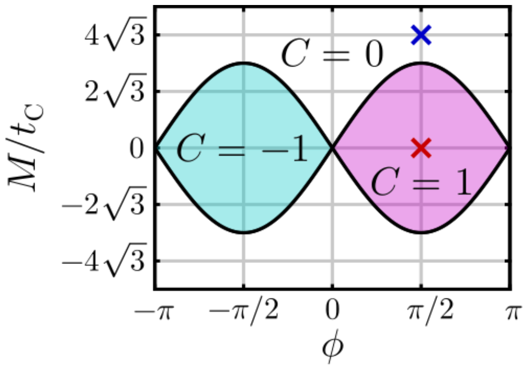


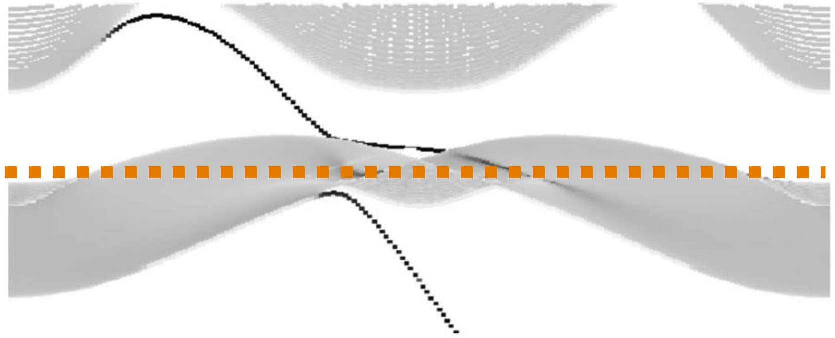
D. Hsieh et al., *Science* **323**, 919 (2009)
Bergman and Refael, *Phys. Rev. B* **82**, 195417 (2010)
Junck et al., *Phys. Rev. B* **87**, 235114 (2013)

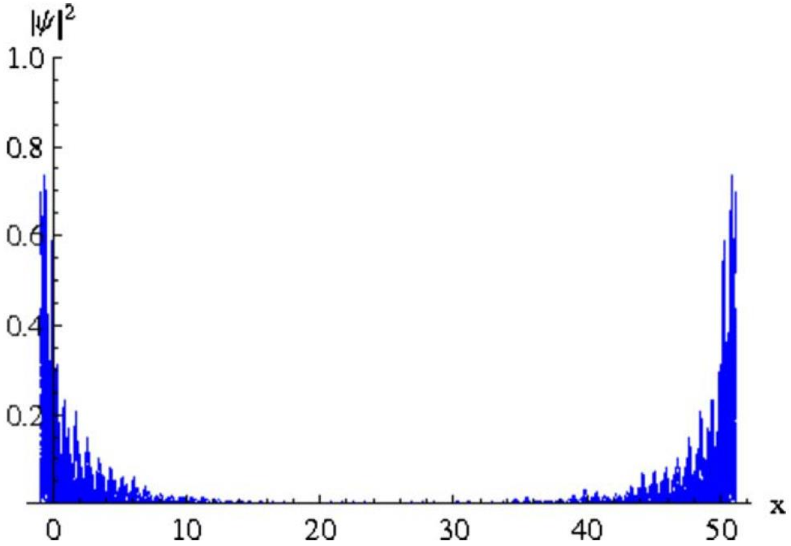
Topology for a gapless system?

“metallized Haldane model”









Found boundary-localized states

- Resistant to hybridization
- Robust against mild disorder

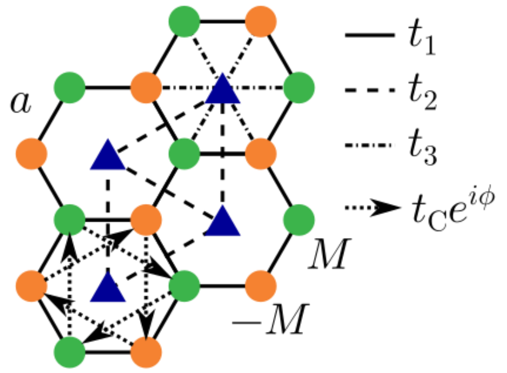
Can such systems be classified?

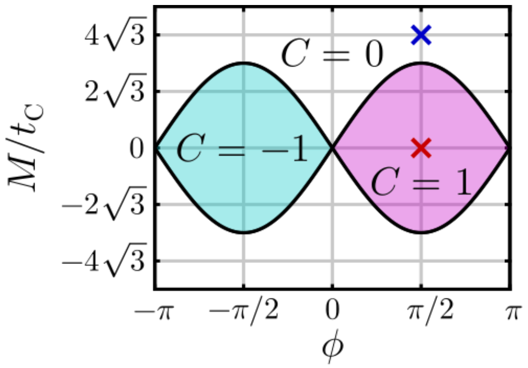
- What if Fermi energy is **here**?
- Measure of topological protection without a band gap?

D. Hsieh et al., *Science* **323**, 919 (2009)
Bergman and Refael, *Phys. Rev. B* **82**, 195417 (2010)
Junck et al., *Phys. Rev. B* **87**, 235114 (2013)

Chern metal

"metallized Haldane model"





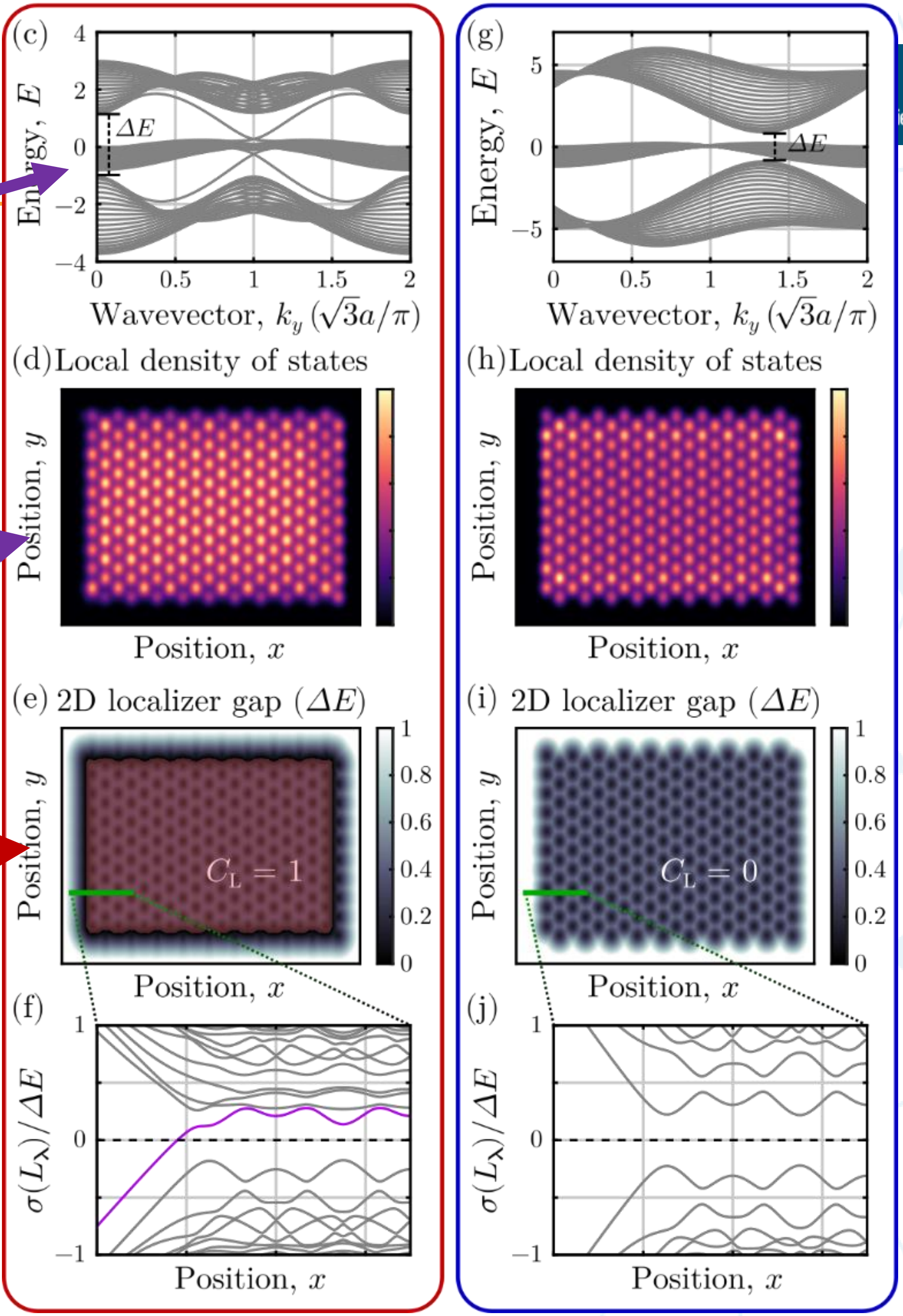
Ribbon band structure

No qualitative difference in LDOS at $E = 0$

Can classify using the spectral localizer

Even though $H - E_F I$ has eigenvalues at 0

$L_{(x,y,E)}$ can still be gapped!



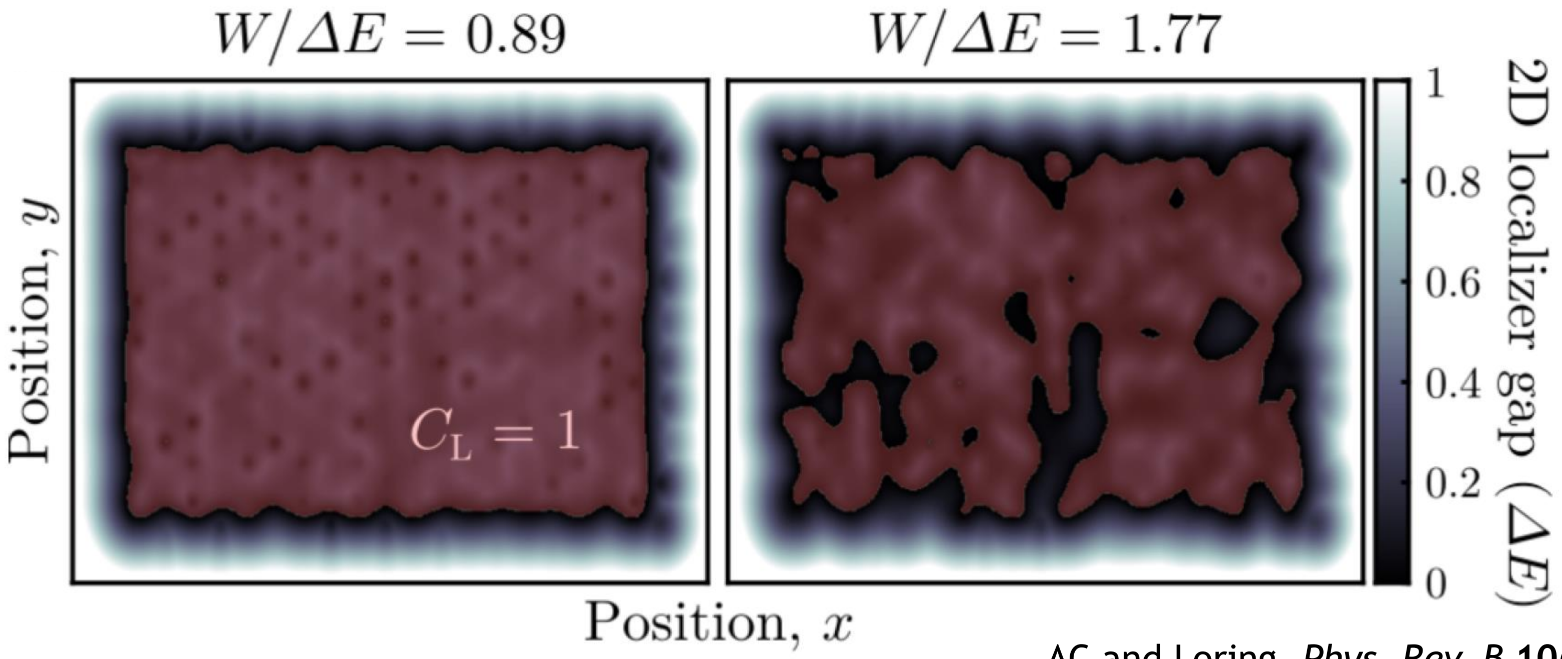
Bergman and Refael, *Phys. Rev. B* **82**, 195417 (2010)
Junck et al., *Phys. Rev. B* **87**, 235114 (2013)

AC and Loring, *Phys. Rev. B* **106**, 064109 (2022)

Disordered Chern metal

By retaining position information from X, Y:

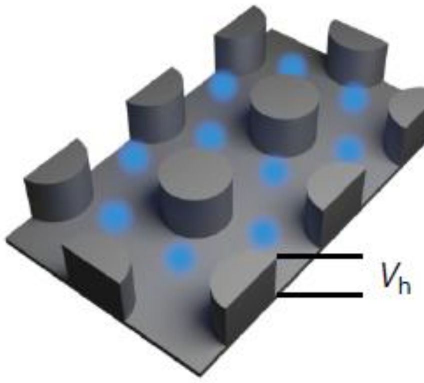
- ➢ Identify local gaps
- ➢ Classify local topology

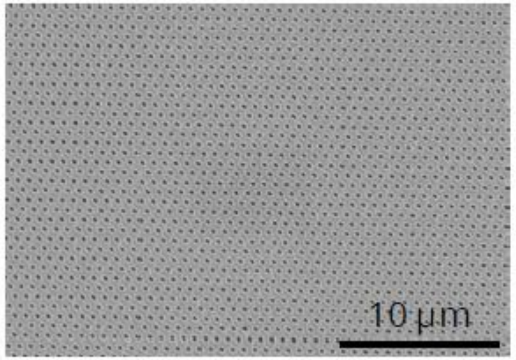


AC and Loring, *Phys. Rev. B* **106**, 064109 (2022)

Application to 2D electron gasses and artificial graphene

Artificial Graphene –
quantum well with added potential V_h



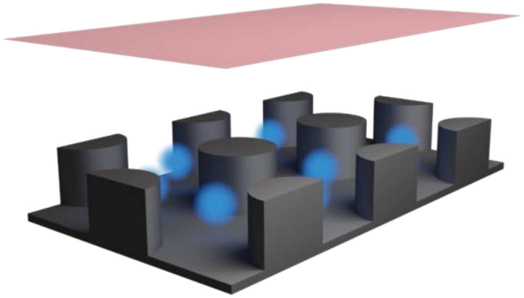


AlSb/InAs/AlSb

$$H = \frac{1}{2m^*}\left(-i\hbar\nabla + e\mathbf{A}(\mathbf{x})\right)^2 + V(\mathbf{x}) - \frac{\mu_B g}{\hbar} s_z B$$

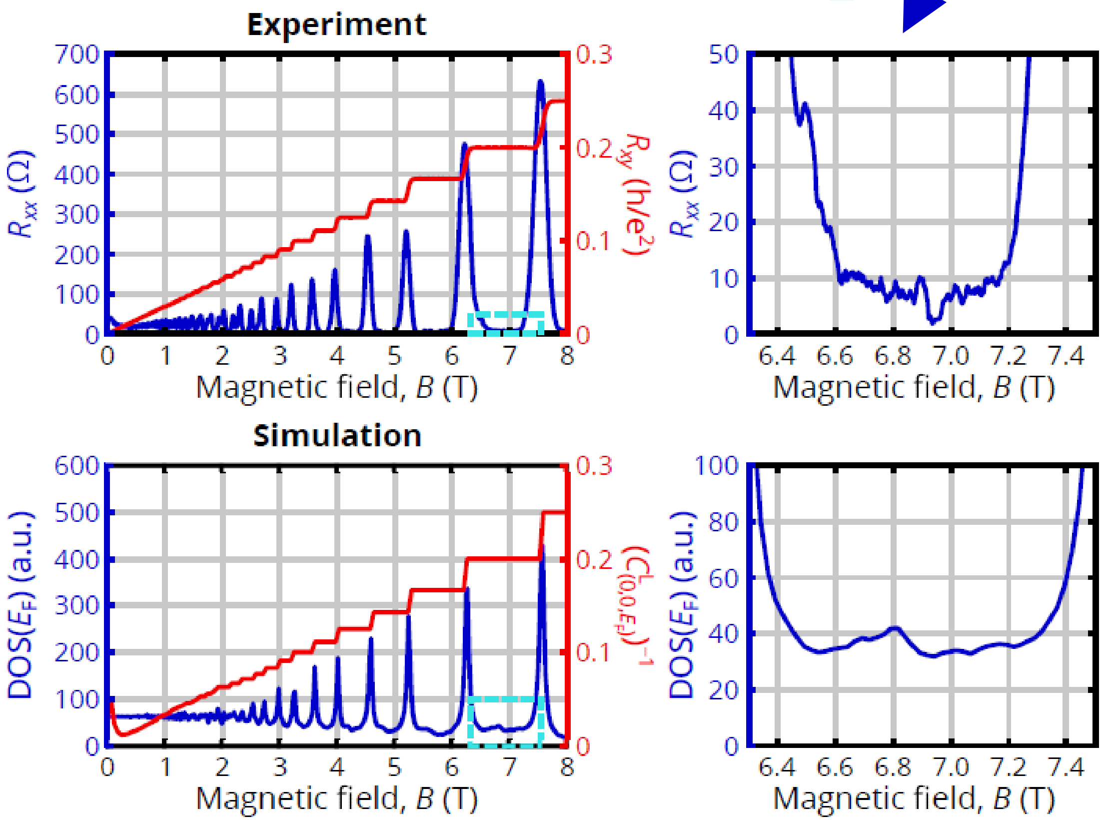
$E_F \approx 4V_h$

- System mostly behaves as 2D electron gas
    - IQHE



Added potential closes the Landau level gaps

Nevertheless, spectral localizer yields correct Hall resistivity



Park et al., *Nano Lett.* **8**, 2920 (2008)
Wunsch et al., *New J. Phys.* **10**, 103027 (2008)

Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Topological origins of pinned states

$L_{(x,y,E)}(X, Y, H)$
$= \kappa(X - xI) \otimes \sigma_x + \kappa(Y - yI) \otimes \sigma_y + (H - EI) \otimes \sigma_z$

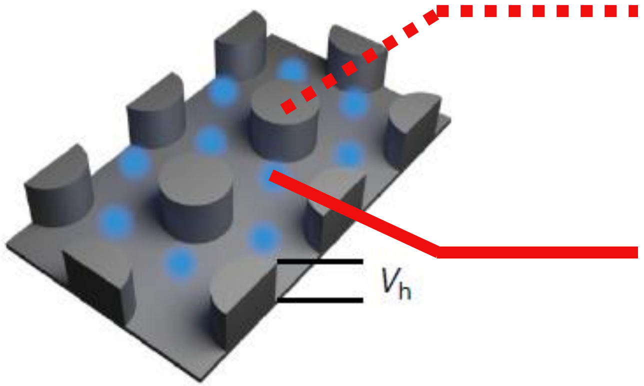
To probe system-level phenomena characterized by length L

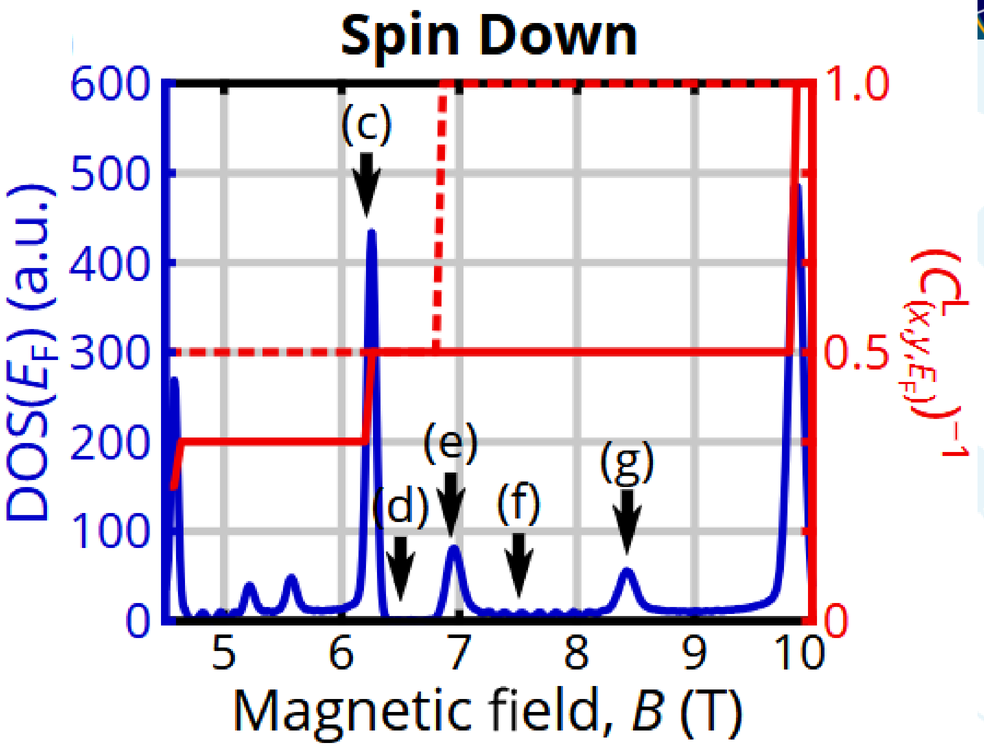
$$\kappa \sim \frac{E_{\text{gap}}}{L}$$

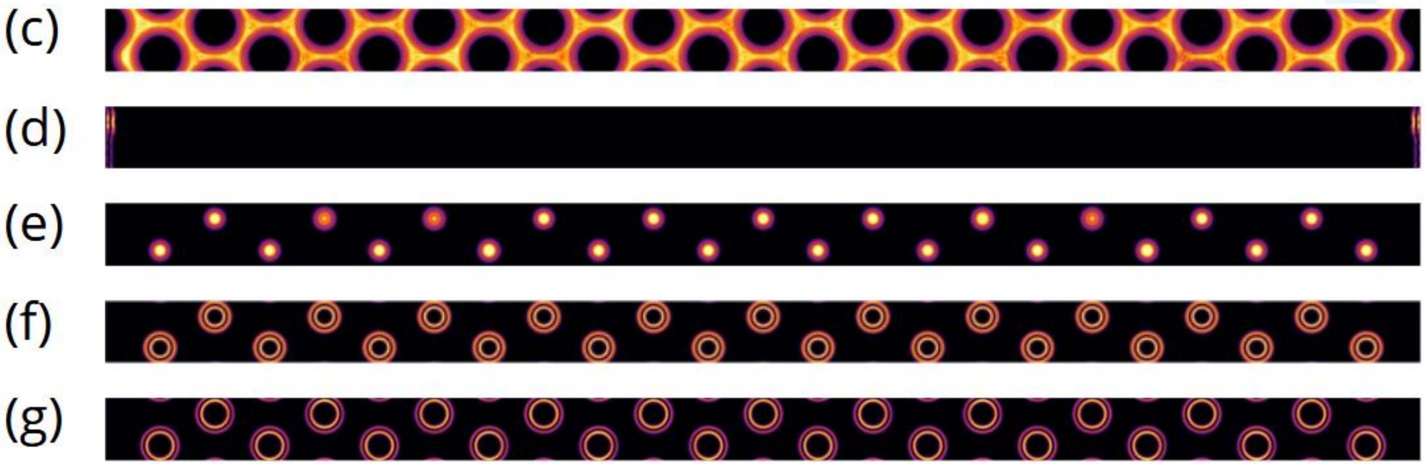
To probe antidot phenomena with diameter D

$$\kappa \sim \frac{E_{\text{gap}}}{D}$$

trading spectral resolution for spatial resolution
→ requires a larger E_{gap}

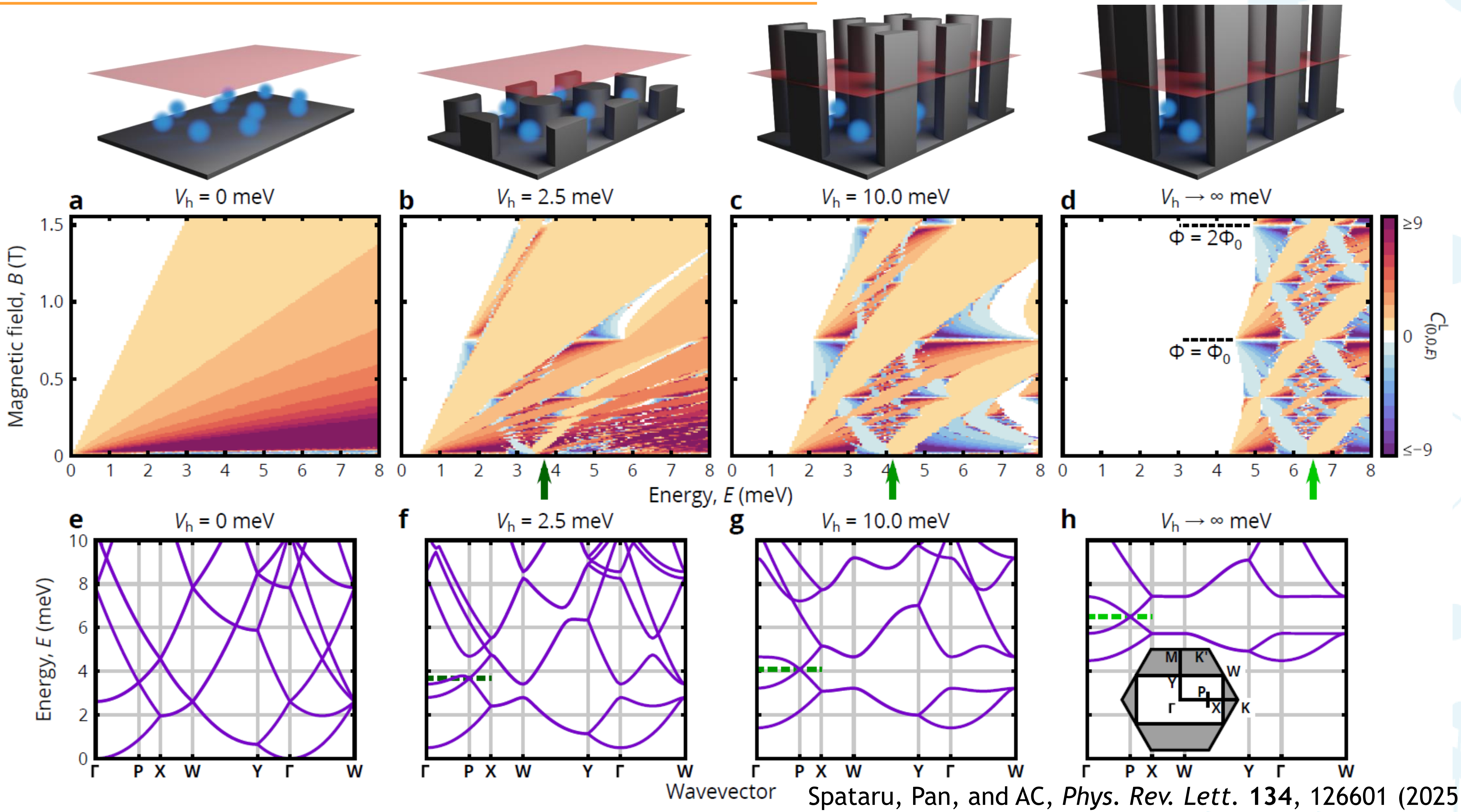






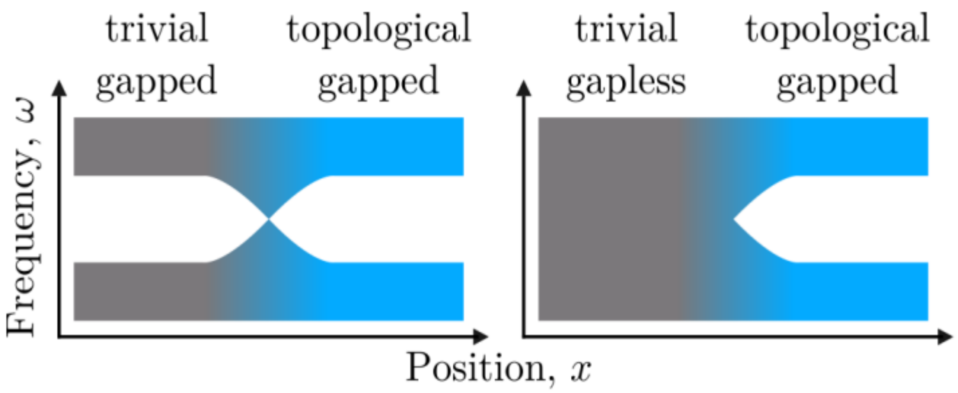
Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Emergence of Hofstader's butterfly as potential is turned on



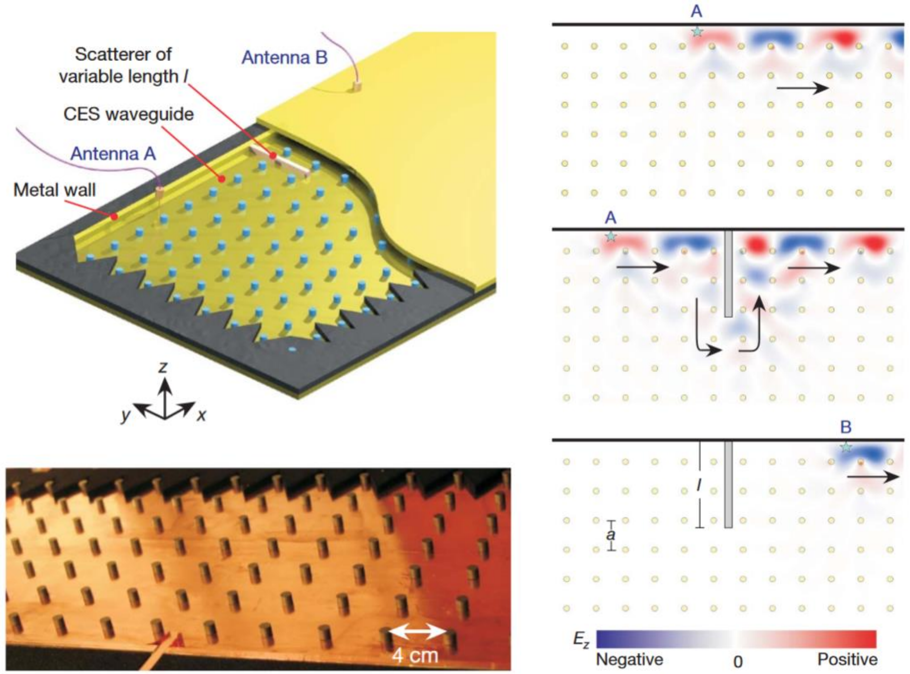
Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)

Radiative environments

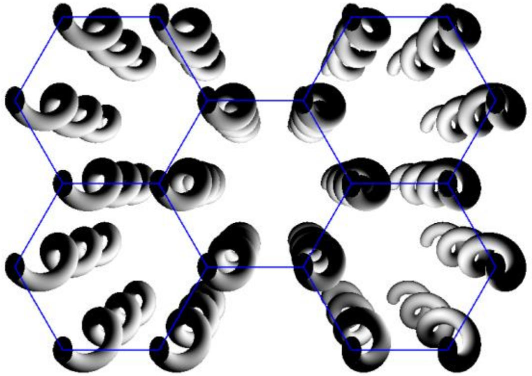


Realized in microwaves

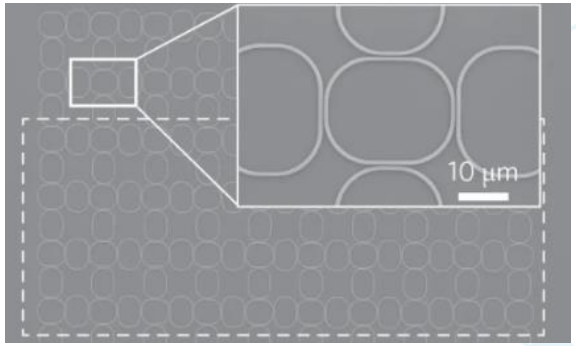
- ➢ Surrounded by a metal
  - ➢ Acts as perfect electric conductor



Wang et. al., *Nature* (2009)



Rechtsman et al., *Nature* (2013)



Hafezi et al., *Nat. Photon.* (2013)

Later realizations in other platforms

- ➢ Surrounded by air
  - ➢ Subject to bending loss

i.e., radiation

Any topological protection against environment perturbations?

Reformulating Maxwell's equations

Linear, local media, allow for dispersion

$$\nabla \times \mathbf{E}(\mathbf{x}) = i\omega\bar{\mu}(\mathbf{x},\omega)\mathbf{H}(\mathbf{x})$$
$$\nabla \times \mathbf{H}(\mathbf{x}) = -i\omega\bar{\varepsilon}(\mathbf{x},\omega)\mathbf{E}(\mathbf{x})$$
$$\nabla \cdot [\bar{\varepsilon}(\mathbf{x},\omega)\mathbf{E}(\mathbf{x})] = 0$$
$$\nabla \cdot [\bar{\mu}(\mathbf{x},\omega)\mathbf{H}(\mathbf{x})] = 0$$

For non-zero frequencies, can recast as:

$$\left[\begin{pmatrix} & -i\nabla\times \\ i\nabla\times & \end{pmatrix} - \omega\begin{pmatrix}\bar{\mu}(\mathbf{x},\omega) & \\ & \bar{\varepsilon}(\mathbf{x},\omega)\end{pmatrix}\right]\begin{pmatrix}\mathbf{H}(\mathbf{x}) \\ \mathbf{E}(\mathbf{x})\end{pmatrix} = 0$$

The divergence equations can be recovered using $\nabla \cdot \nabla \times \mathbf{F}(\mathbf{x}) = 0$ for any vector field $\mathbf{F}(\mathbf{x})$, for any $\omega \neq 0$

This yields a "self-consistent" generalized eigenvalue equation:

$$\boxed{W\boldsymbol{\psi}(\mathbf{x}) = \omega M(\mathbf{x},\omega)\boldsymbol{\psi}(\mathbf{x})}$$

$$W = \begin{pmatrix} & -i\nabla\times \\ i\nabla\times & \end{pmatrix} \qquad \boldsymbol{\psi}(\mathbf{x}) = \begin{pmatrix}\mathbf{H}(\mathbf{x}) \\ \mathbf{E}(\mathbf{x})\end{pmatrix}$$

$$M(\mathbf{x},\omega) = \begin{pmatrix}\bar{\mu}(\mathbf{x},\omega) & \\ & \bar{\varepsilon}(\mathbf{x},\omega)\end{pmatrix}$$

And finally an ordinary eigenvalue equation:

$$\boxed{H\boldsymbol{\phi}(\mathbf{x}) = \omega\boldsymbol{\phi}(\mathbf{x})}$$

$$H = M^{-1/2}(\mathbf{x},\omega)WM^{-1/2}(\mathbf{x},\omega)$$

$$\boldsymbol{\phi}(\mathbf{x}) = M^{1/2}(\mathbf{x},\omega)\boldsymbol{\psi}(\mathbf{x})$$

Radiative environments

For Chern number, non-Hermitian generalization is known

$$L_{(x,y,\omega)}(X,Y,H) = \begin{bmatrix} H-\omega I & \kappa(X-xI)-i\kappa(Y-yI) \\ \kappa(X-xI)+i\kappa(Y-yI) & -(H-\omega I)^{\dagger} \end{bmatrix}$$

Yielding

$$C^{L}_{(x,y,E)} = \frac{1}{2}\mathrm{sig}[L_{(x,y,E)}(X,Y,H)] \in \mathbb{Z}$$

(signature now counts positive real parts minus negative real parts)

AC, Koekenbier, and Schulz-Baldes, *J. Math. Phys.* **64**, 082102 (2023)

Kahlil Y. Dixon

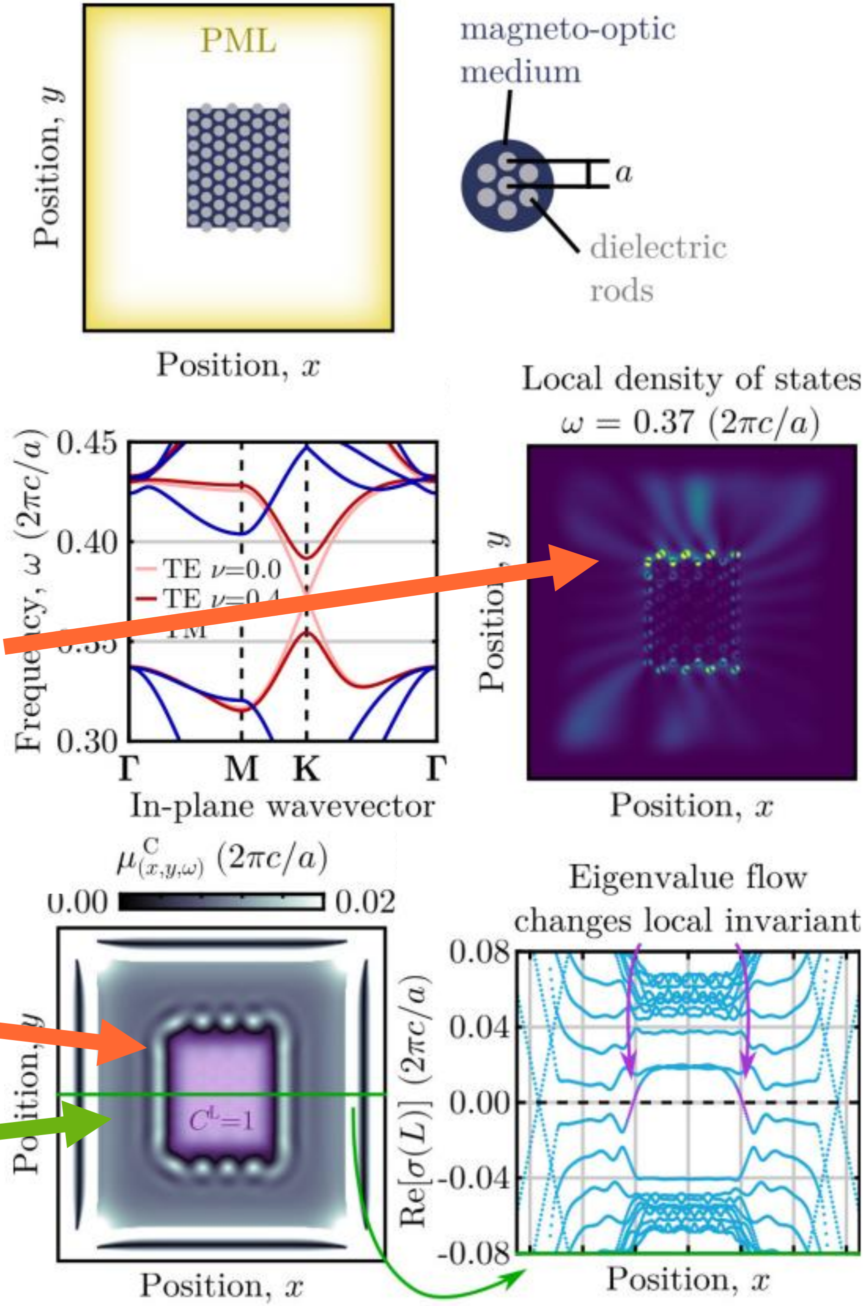
Non-zero local gap!

- Topological protection against perturbations in the environment!

LDOS shows a chiral edge resonance

Spectral localizer proves existence of chiral edge resonance

- Resonance… not state.
- Couples to vacuum.



Dixon, Loring, and AC, *Phys. Rev. Lett.* **131**, 213801 (2023)

General framework for non-linear topology

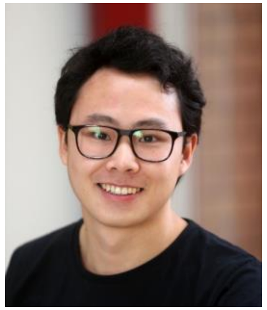
Working in real-space

- Can handle spatial non-linearities **for free**

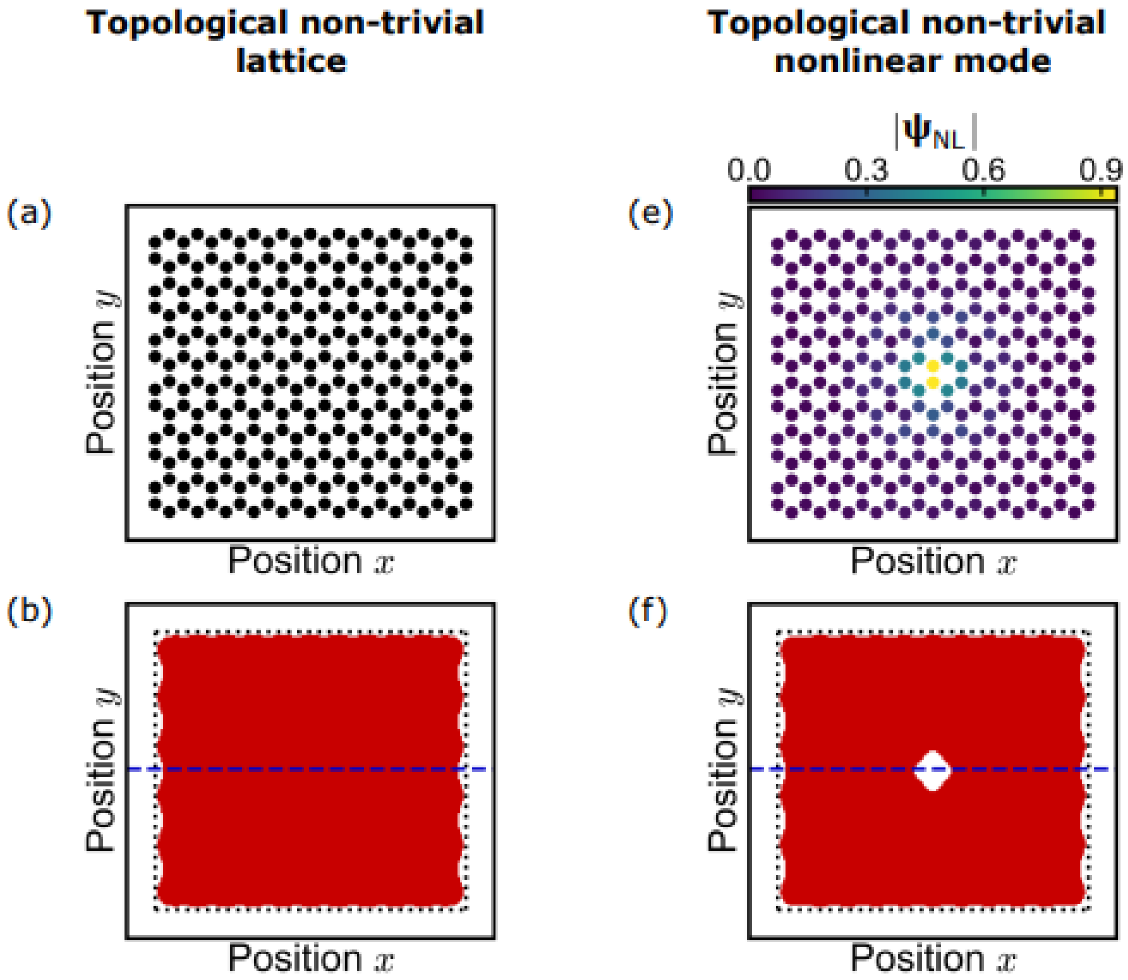
$$L_{(x,y,E)}(X,Y,H_{\text{NL}}(\boldsymbol{\psi})) = \begin{bmatrix} H_{\text{NL}}(\boldsymbol{\psi}) - EI & \kappa(X-xI) - i\kappa(Y-yI) \\ \kappa(X-xI) + i\kappa(Y-yI) & -(H_{\text{NL}}(\boldsymbol{\psi}) - EI) \end{bmatrix}$$

On-site non-linearity

$$H_{\text{NL}}(\boldsymbol{\psi}) = H_0 + g|\boldsymbol{\psi}|^2$$



Stephan Wong



Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

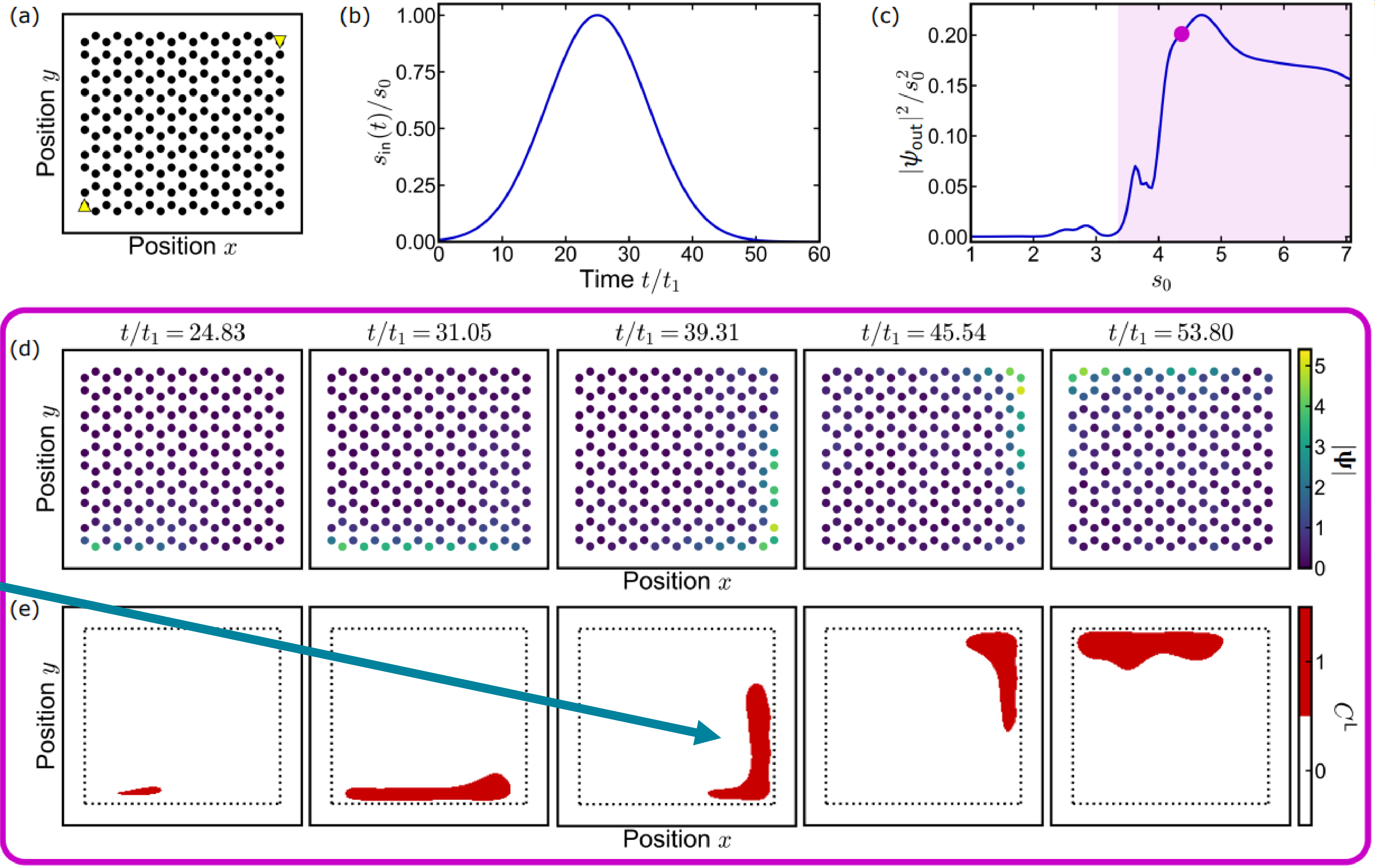
Topological dynamics

Previously predicted and observed edge solitons

Leykam and Chong, *Phys. Rev. Lett.* **117**, 143901 (2016)

Mukherjee and Rechtsman, *Phys. Rev. X* **11**, 041057 (2021)

Non-linear topological dynamics!



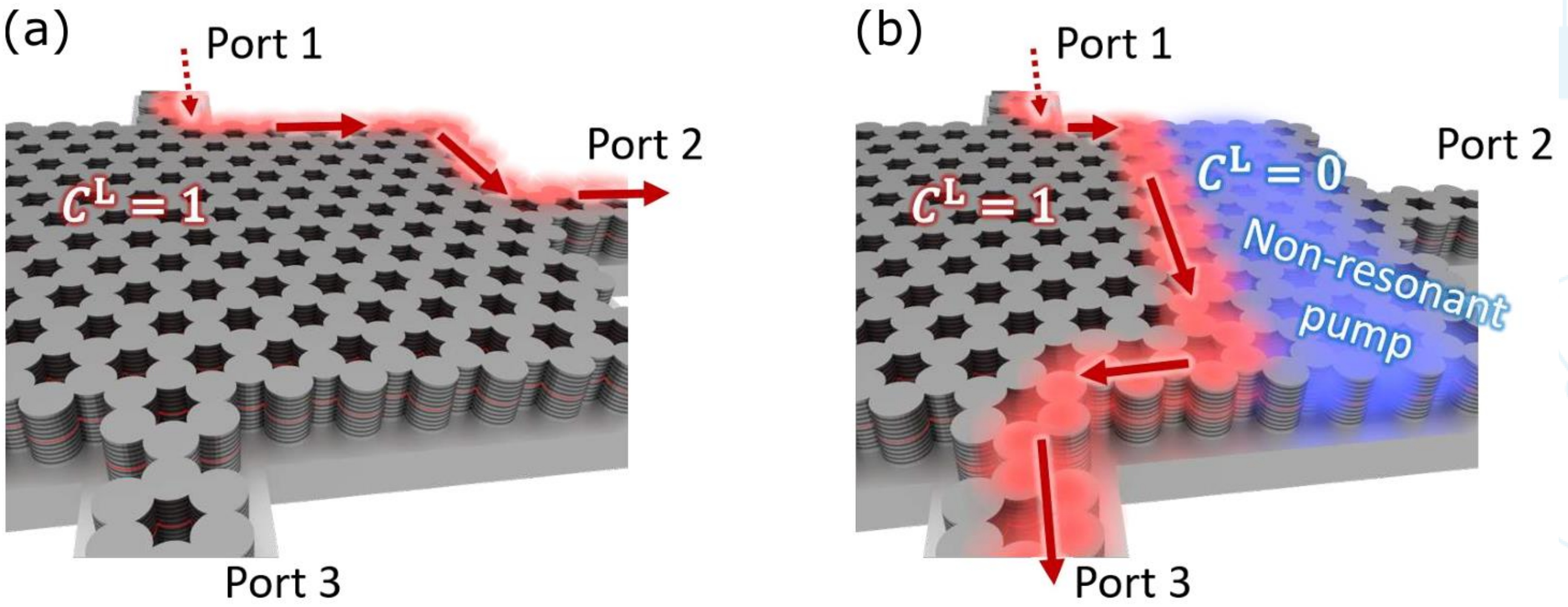
Wong, Loring, and AC, *Phys. Rev. B* **108**, 195142 (2023)

Reconfigurable topology in exciton-polariton lattices

Fast reconfiguration for protected routing

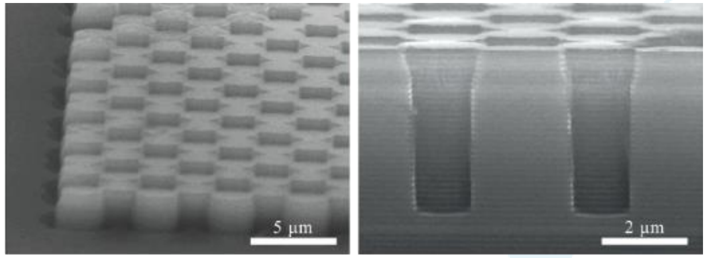
- ❖ GHz+ speeds

➢ Need a local framework for driven-dissipative systems



Driven-dissipative exciton-polariton systems

$$i\hbar \frac{\partial}{\partial t}\psi = H_0\psi - i\hbar\left(\frac{\gamma_c}{2}\right)\psi + g_c|\psi|^2\psi + \left(g_r + i\hbar\frac{R}{2}\right)n_r\psi + S_{probe}$$
$$\frac{\partial}{\partial t}n_r = -(\gamma_r + R|\psi|^2)n_r + S_{pump}$$



Parameters from

Klembt et al., *Nature* **562**, 552 (2018)

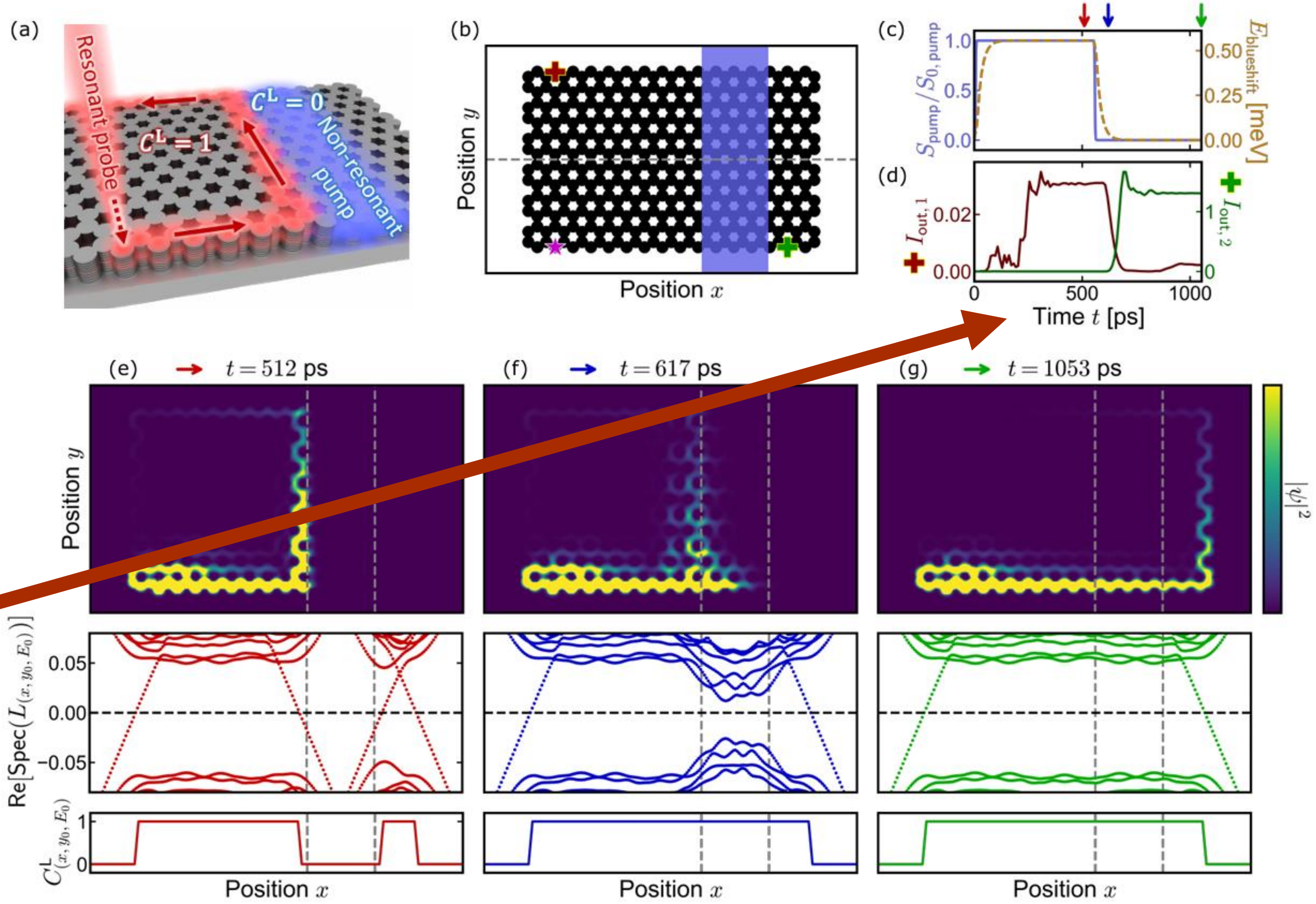
Wong, Betzold, Höfling, AC, in press at *Light: Science & Applications*

Reconfigurable topology in exciton-polariton lattices

Spectral localizer for driven-dissipative systems

$$L_{(x,y,\omega)}\left(X,Y,H_{\mathrm{NL}}(\boldsymbol{\psi}(t))\right) = \begin{bmatrix} H_{\mathrm{NL}}(\boldsymbol{\psi}(t)) - \omega I & \kappa(X - xI) - i\kappa(Y - yI) \\ \kappa(X - xI) + i\kappa(Y - yI) & -\left(H_{\mathrm{NL}}(\boldsymbol{\psi}(t)) - \omega I\right)^{\dagger} \end{bmatrix}$$

Full switching cycle completed in 1ns!

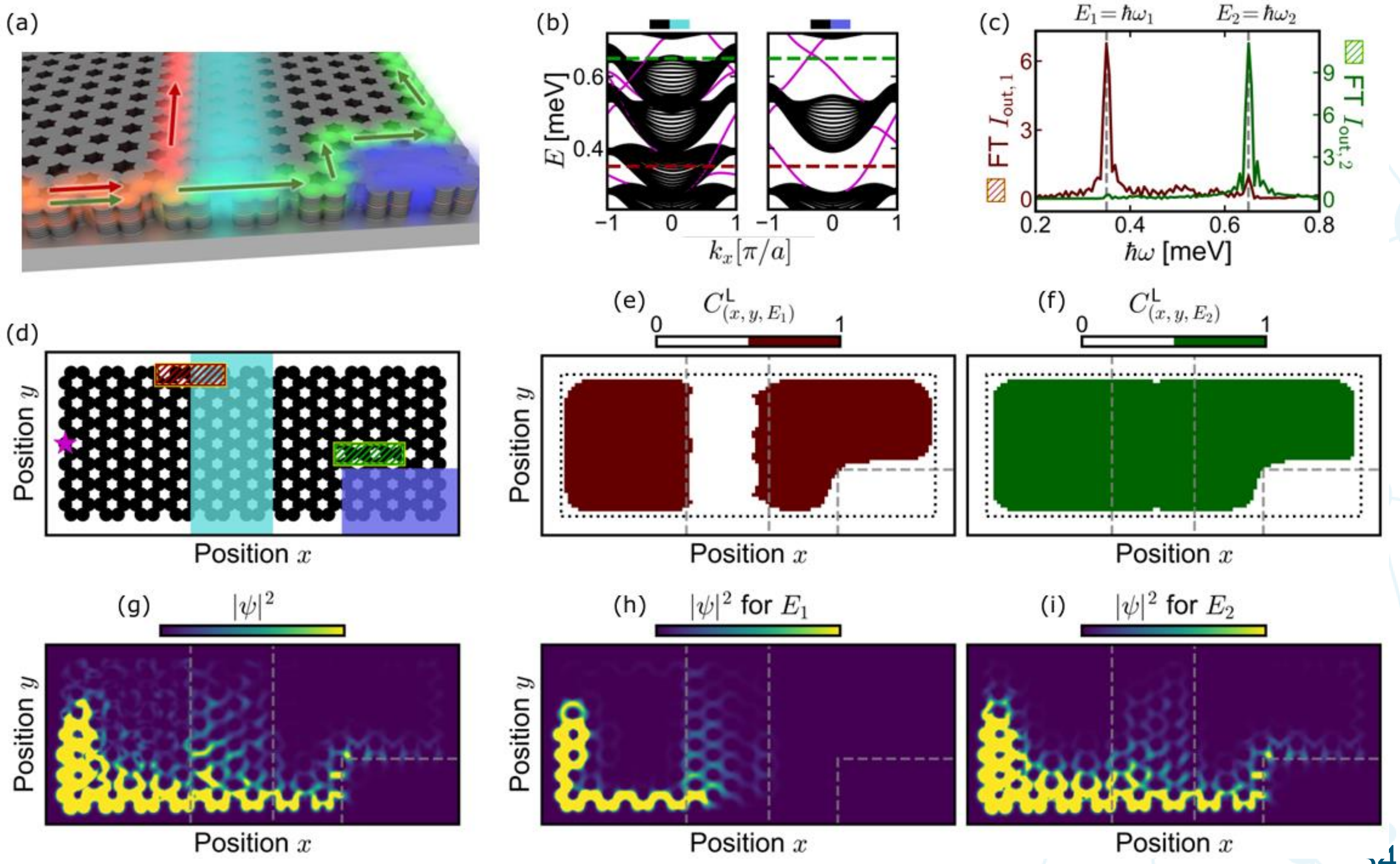


Wong, Betzold, Höfling, AC, in press at *Light: Science & Applications*

Multi-channel reconfigurable topology

Local topology seen by **lower** and **higher** frequency signals is different

- Each follows a different edge path



Wong, Betzold, Höfling, AC, in press at *Light: Science & Applications*

Local winding number (1D class AIII)

In 1D, the spectral localizer is

$$L_{(x,E)}(X,H) = \kappa(X - xI) \otimes \sigma_x + (H - EI) \otimes \sigma_y = \begin{bmatrix} 0 & \kappa(X - xI) - i(H - EI) \\ \kappa(X - xI) + i(H - EI) & 0 \end{bmatrix}$$

These two blocks contain the same spectral information…

… Instead work with a single block

$$\kappa(X - xI) - i(H - EI)$$

But, this is not Hermitian.

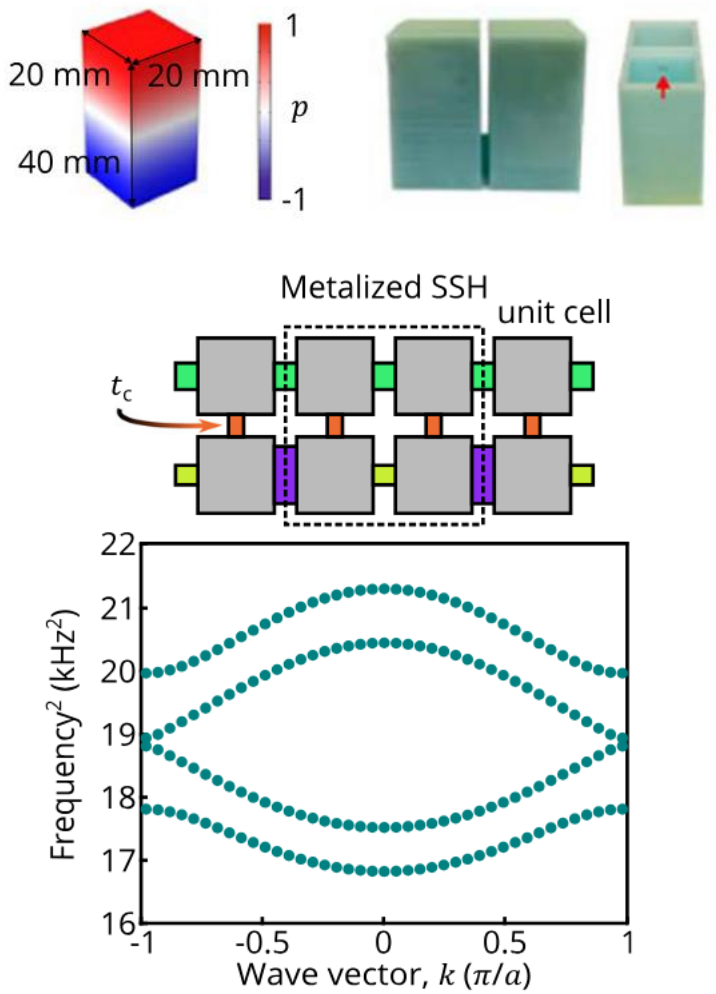
Use the system's chiral symmetry to restore Hermiticity, $\Pi H = -H\Pi$

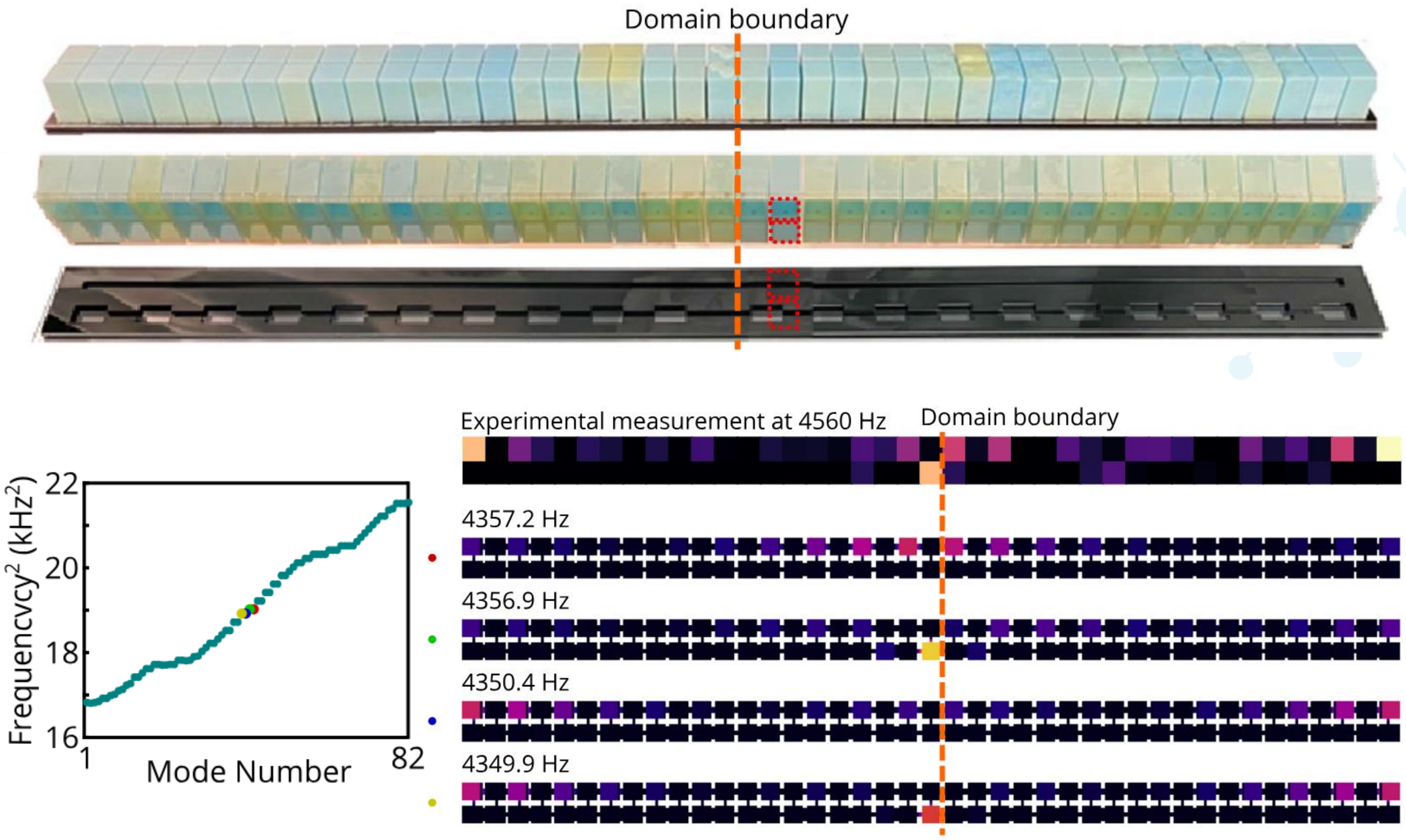
$$\tilde{L}_x(X,H) = [\kappa(X - xI) - iH]\Pi$$

⇒ Local winding number

$$\nu_x^L = \frac{1}{2}\mathrm{sig}[\tilde{L}_x(X,H)] \in \mathbb{Z}$$

Gapless 1D system





W. Cheng*, AC*, S.-Y. Chen, E. Prodan, T. A. Loring, and C. Prodan, *Nat. Commun.* **14**, 3071 (2023)

Operators don't care about physical meaning

In 1D class AIII (e.g., SSH model), chiral symmetry protects states at $E = 0$

$$H\Pi = -\Pi H, \quad X\Pi = \Pi X, \quad \Pi^2 = I, \quad \Pi = \Pi^\dagger$$

Local winding number:

$$\upsilon_x = \frac{1}{2}\mathrm{sig}[(\kappa(X - xI) + iH)\Pi] \in \mathbb{Z}$$

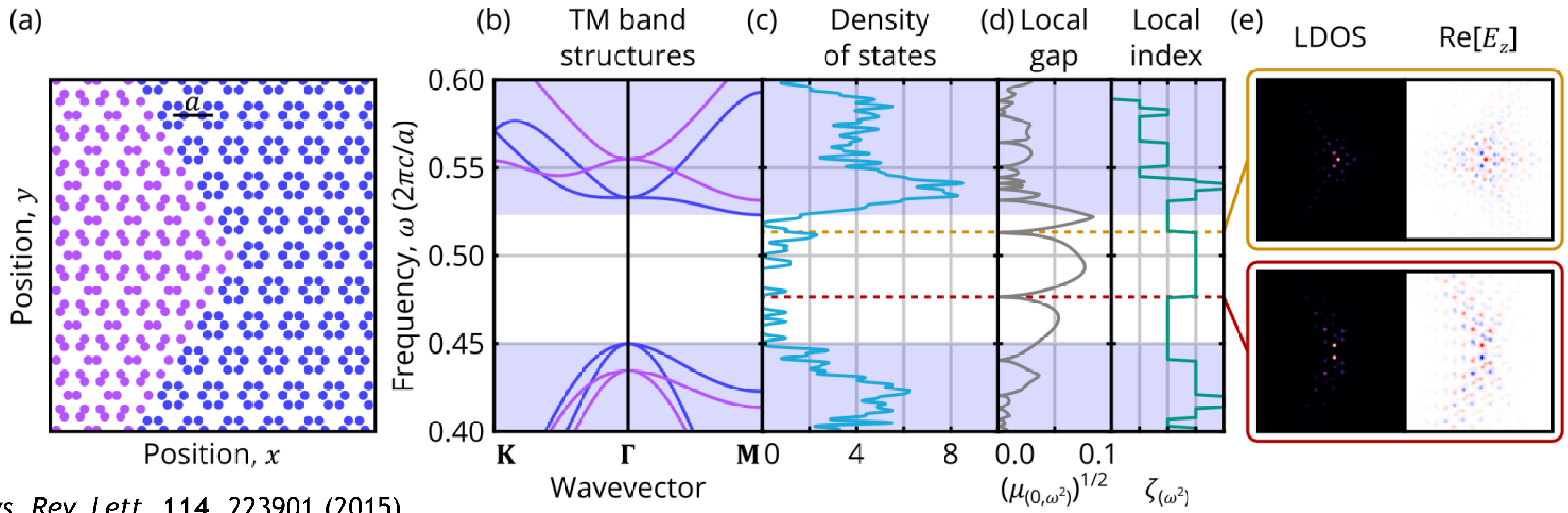
But crystalline symmetry can yield similar commutation relations

$$H\mathcal{S} = \mathcal{S}H, \quad X\mathcal{S} = -\mathcal{S}X, \quad \mathcal{S}^2 = I, \quad \mathcal{S} = \mathcal{S}^\dagger$$

Local "crystalline winding number," protects states at $x = 0$

$$\zeta_E^{\mathcal{S}} = \frac{1}{2}\mathrm{sig}[(H - EI + i\kappa X)\mathcal{S}] \in \mathbb{Z}$$

Local markers for crystalline topology



Wu, Hu, *Phys. Rev. Lett.* **114**, 223901 (2015)
Smirnova et al., *Phys. Rev. Lett.*, **123**, 103901 (2019)
Kruk et al., *Nano Lett.* **21**, 4592 (2021)

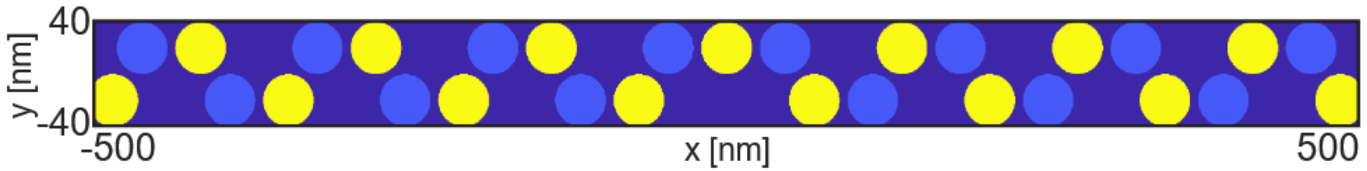
Local "reflection winding number," protects states at $y = 0$

$$\zeta_{\omega}^{\mathcal{R}_y} = \frac{1}{2}\,\mathrm{sig}\big[(H - \omega I + i\kappa Y)\mathcal{R}_y\big] \in \mathbb{Z}$$

AC, Loring, and Schulz-Baldes, *Phys. Rev. Lett.* **132**, 073803 (2024)

Valley-Hall effect from reflection topology

Artificial hBN in a 2D electron gas



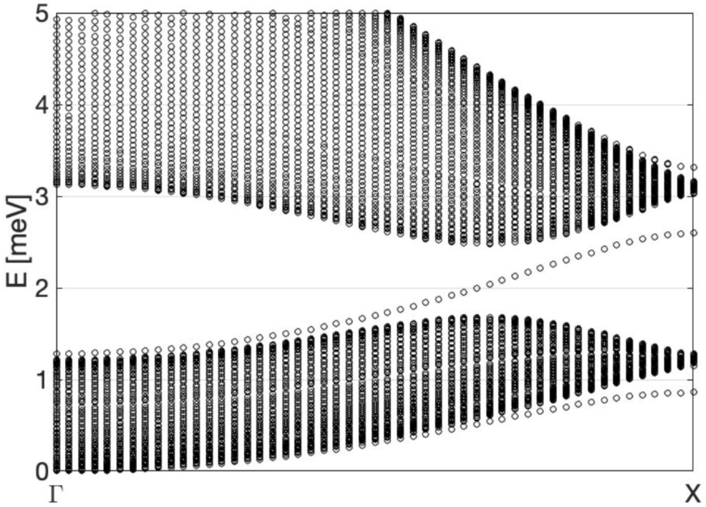
Identify the valley-Hall effect using a reflection symmetry

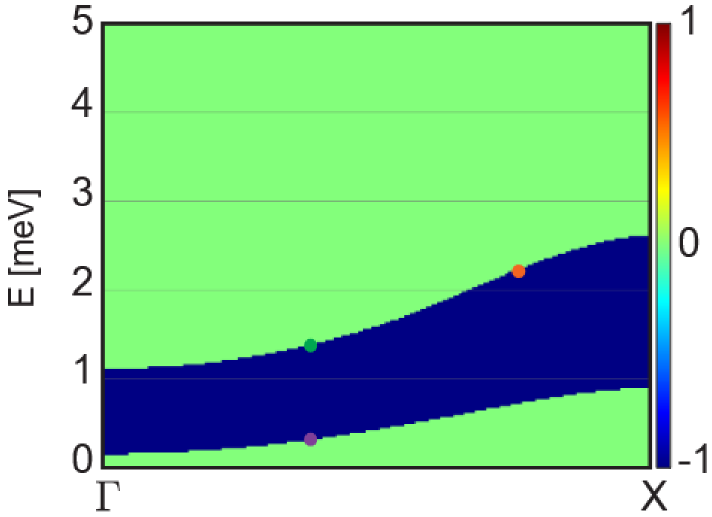
$\Rightarrow$ Can keep good wavevector k_y

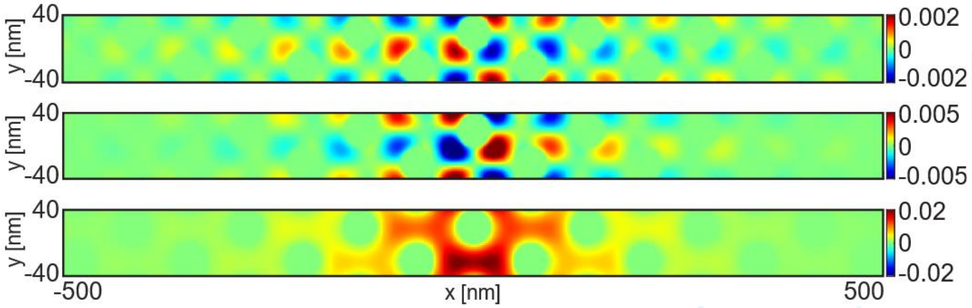
$$\tilde{L}_E\left(X, H(k_y)\right) = [H(k_y) - EI + i\kappa X]\mathcal{R}$$

Yielding the invariant

$$\zeta^{\mathcal{R}}_{(E,k_y)} = \frac{1}{2}\operatorname{sig}[(H(k_y) - EI + i\kappa X)\mathcal{R}] \in \mathbb{Z}$$







Pai, Cummings, AC, Pan, Zhang, and Spataru, arXiv:2510.05250

Topological dimensional crossover

How does a system's topology change as it changes dimension?

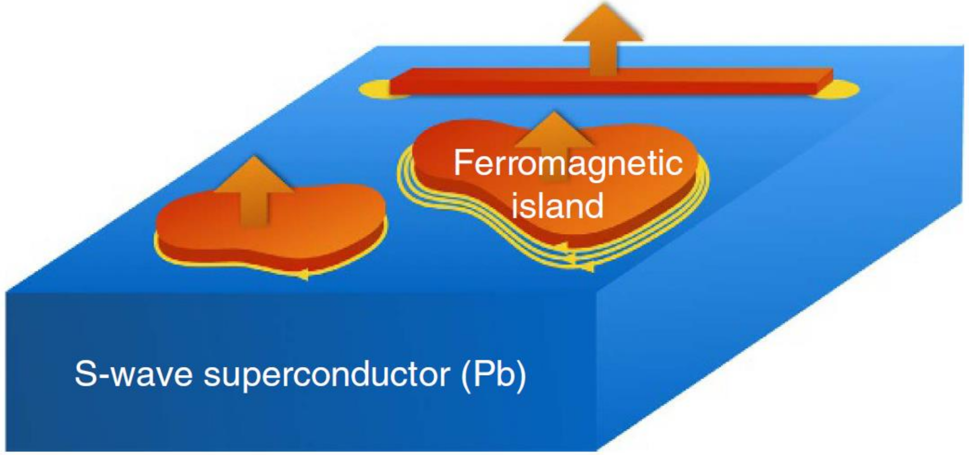
Study with topological superconductors (class D)

- $\mathbb{Z}_2$ in 1D
- $\mathbb{Z}$ in 2D

$$H = -t \sum_{\mathbf{r},\boldsymbol{\delta},\sigma} c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}+\boldsymbol{\delta}\sigma} - \mu \sum_{\mathbf{r},\sigma} c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}\sigma} + \Delta \sum_{\mathbf{r}} c^\dagger_{\mathbf{r}\uparrow} c^\dagger_{\mathbf{r}\downarrow} + h.c.$$
$$+ i\alpha \sum_{\mathbf{r},\boldsymbol{\delta},a,b} c^\dagger_{\mathbf{r}a} \left(\boldsymbol{\delta} \times \boldsymbol{\sigma}\right)^z_{ab} c_{\mathbf{r}+\boldsymbol{\delta}b} + J \sum_{\mathbf{R},a,b} \mathbf{S} \cdot c^\dagger_{\mathbf{R}a} \boldsymbol{\sigma}_{ab} c_{\mathbf{R}b},$$

Shiba Lattice –

Magnetic adatoms on a conventional superconductor



Li et al., *Nat. Commun.* **7**, 12297 (2016)

1D local marker given by

$$L_{(x,E)}(X,H) = \kappa(X - xI) \otimes \sigma_x + (H - EI) \otimes \sigma_y$$

$\mathcal{C}$ symmetry allows basis where $H^\top = -H$

$\Rightarrow \kappa(X - xI) + iH$ is real

$$z_{2,x} = \mathrm{sign}[\det[\kappa(X - xI) + iH]] \in \mathbb{Z}_2$$

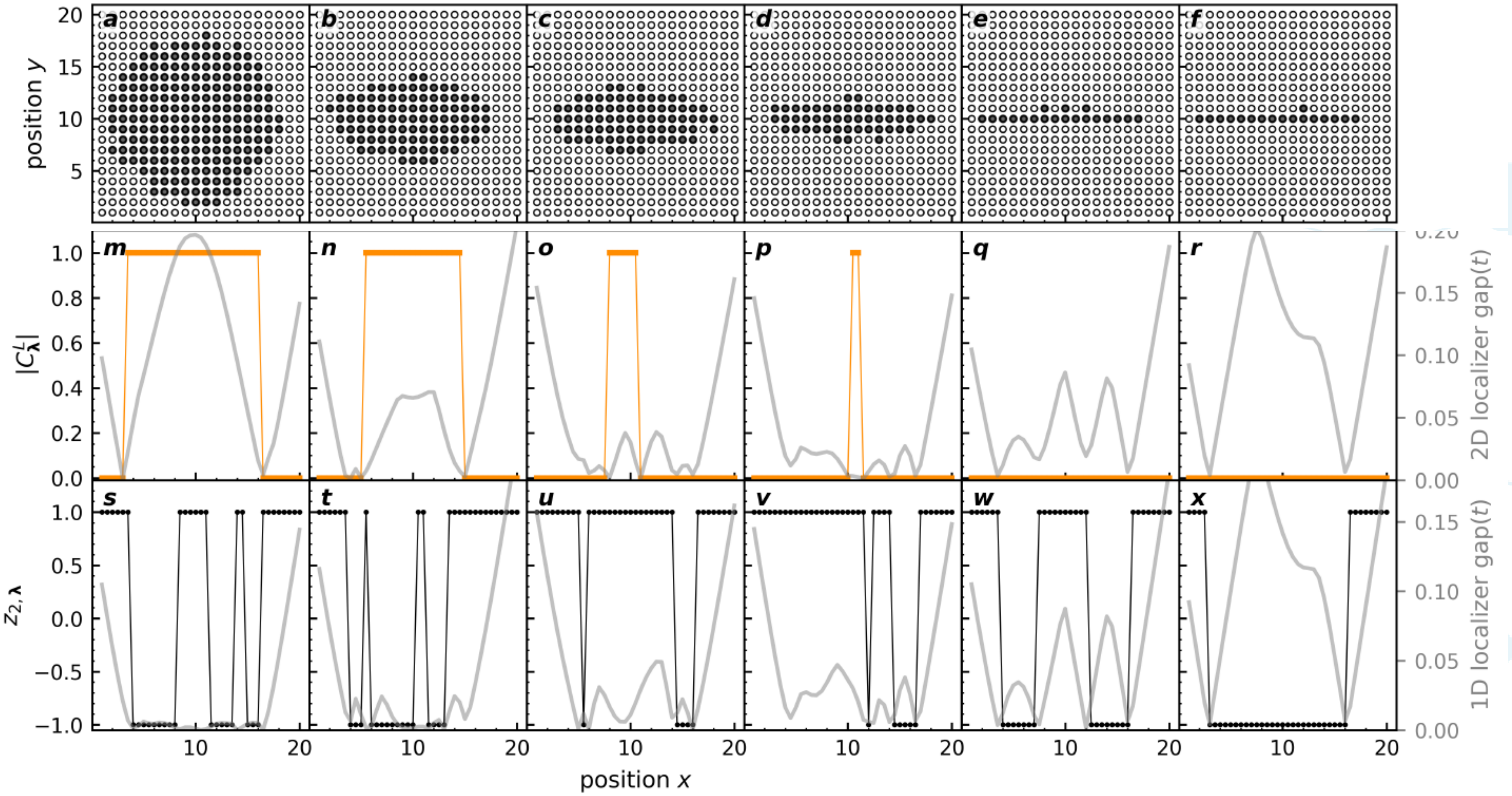
(Still use same H, just "forget" about a position operator)

Topological dimensional crossover

Change the distribution o
magnetic adatoms

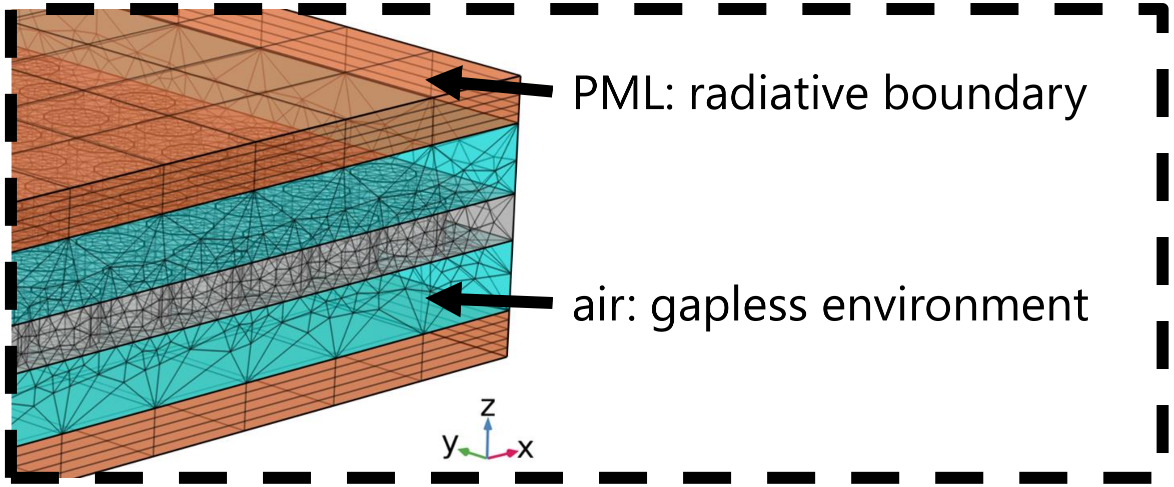
- Directly monitor the change in topology
- And its robustness

Even with disorder!



Rodriguez-Vega, Loring, and AC, in press at *Commun. Phys.*

Topology in Photonic Crystal Slabs

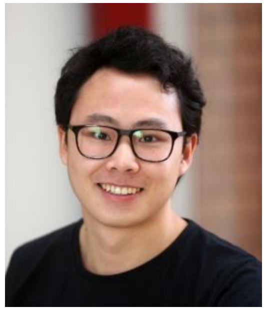


class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

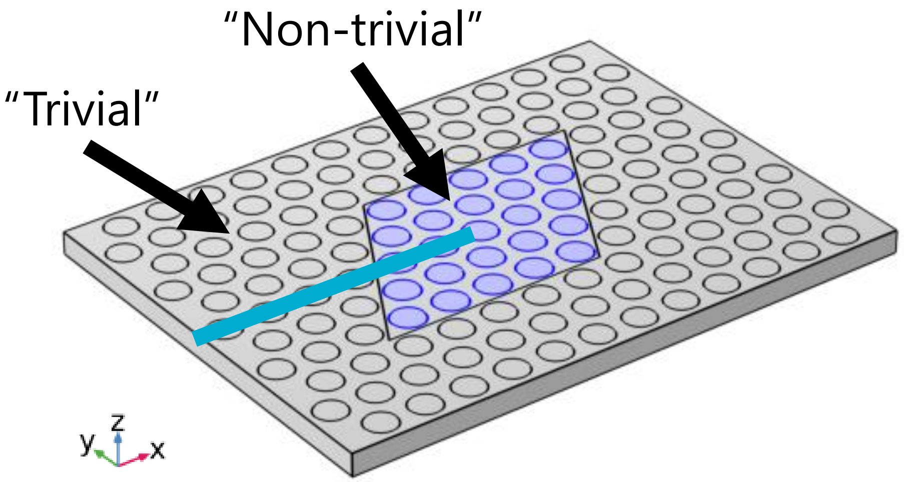
Schnyder *et. al.*, *Phys. Rev. B* **78**, 195125 (2008)

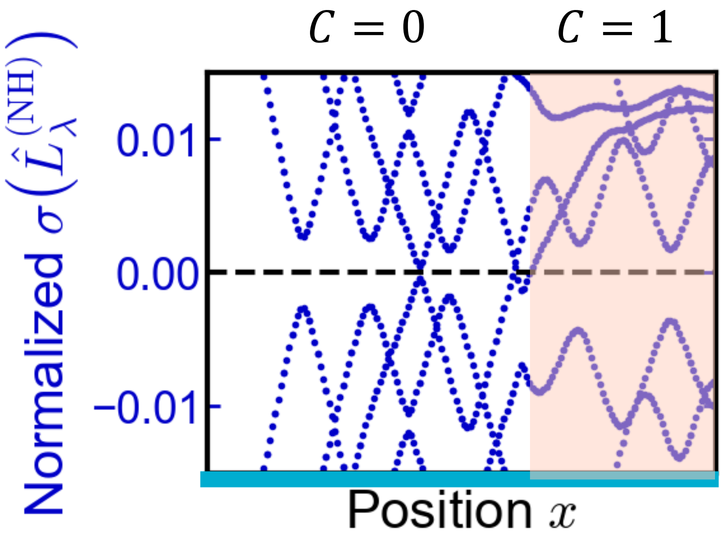
Topological edge states in slab with 2D strong topological invariant

- Disregard z-direction: $(x, y, z) \to (x, y)$
  (still have all vertices, just "forgetting" about z)
- Look at the change of topology in the (x, y)-plane



Stephan Wong





Wong, Loring, and AC, *npj Nanophoton.* **1**, 19 (2024)

Classifying fragile topology via matrix homotopy

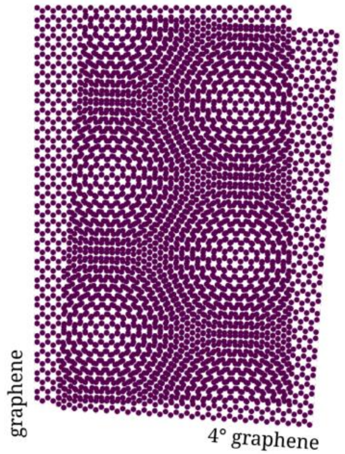
Fragile topology can be protected by $(C_2\mathcal{T})$-symmetry

C_2 – 180° rotation about out-of-plane axis
$\mathcal{T}$ – Bosonic time-reversal symmetry, $\mathcal{T}^2 = I$

For a system with this symmetry

$$(C_2\mathcal{T})^{-1}H(C_2\mathcal{T}) = H$$

$$(C_2\mathcal{T})^{-1}X(C_2\mathcal{T}) = -X$$
$$(C_2\mathcal{T})^{-1}Y(C_2\mathcal{T}) = -Y$$



Ponor, Wikimedia Commons

Define

$$M^\rho = (C_2\mathcal{T})^{-1}M^\dagger(C_2\mathcal{T})$$

ρ is some sort of "symmetry-informed" matrix operation

$$H^\rho = H$$
$$X^\rho = -X$$
$$Y^\rho = -Y$$

Can be simplified to

$$M^\rho = C_2 M^\top C_2$$

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

Classifying fragile topology via matrix homotopy

Theorem:

If ρ commutes with $\dagger$, i.e., if $(M^\rho)^\dagger = (M^\dagger)^\rho$

then there is guaranteed to be a basis in which $\rho \to \top$ i.e., the matrix transpose

Directly verify $(M^\rho)^\dagger = C_2(M^\top)^\dagger C_2 = C_2(M^\dagger)^\top C_2 = (M^\dagger)^\rho$

And for $W = \frac{1}{\sqrt{2}}(C_2 + iI)$

$$\begin{aligned} H^\rho &= H && \longrightarrow && (WHW^\dagger)^\top = WHW^\dagger \\ X^\rho &= -X && \longrightarrow && (WXW^\dagger)^\top = -WXW^\dagger \\ Y^\rho &= -Y && \longrightarrow && (WYW^\dagger)^\top = -WYW^\dagger \end{aligned}$$

symmetric

skew symmetric

$$M^\rho = (C_2\mathcal{T})^{-1} M^\dagger (C_2\mathcal{T})$$

So, ρ is really a symmetry-informed matrix transpose

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

Classifying fragile topology via matrix homotopy

$$\left.\begin{array}{lcl} H^\rho = H & \longrightarrow & \left(WHW^\dagger\right)^\top = WHW^\dagger \\ X^\rho = -X & \longrightarrow & \left(WXW^\dagger\right)^\top = -WXW^\dagger \\ Y^\rho = -Y & \longrightarrow & \left(WYW^\dagger\right)^\top = -WYW^\dagger \end{array}\right.$$

symmetric

skew symmetric

$$\begin{array}{l} \sigma_x^\top = \sigma_x \\ \sigma_z^\top = \sigma_z \\ \sigma_y^\top = -\sigma_y \end{array}$$

Pair each skew-symmetric with a symmetric

- Forms a (nearly) skew-symmetric spectral localizer

$$L_{(x,y,E)}\left(WXW^\dagger, WYW^\dagger, WHW^\dagger\right) = \kappa\left(WXW^\dagger - x\right) \otimes \sigma_x + \kappa\left(WYW^\dagger - y\right) \otimes \sigma_z + \left(WHW^\dagger - E\right) \otimes \sigma_y$$

$$= \begin{bmatrix} \kappa\left(WYW^\dagger - y\right) & \kappa\left(WXW^\dagger - x\right) - i\left(WHW^\dagger - E\right) \\ \kappa\left(WXW^\dagger - x\right) + i\left(WHW^\dagger - E\right) & -\kappa\left(WYW^\dagger - y\right) \end{bmatrix}$$

At $(x, y) = (0,0)$, $L_{(0,0,E)}^\top = -L_{(0,0,E)}$

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

Homotopy invariant of skew symmetric matrices

$$T = \begin{bmatrix} 0 & \alpha_1 & & & & & & \\ -\alpha_1 & 0 & & & & & & \\ & & 0 & \alpha_2 & & & & \\ & & -\alpha_2 & 0 & & & & \\ & & & & & \ddots & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \alpha_n \\ & & & & & & -\alpha_n & 0 \end{bmatrix}$$

Skew symmetric — $T^{\top} = -T$

Well-defined sign

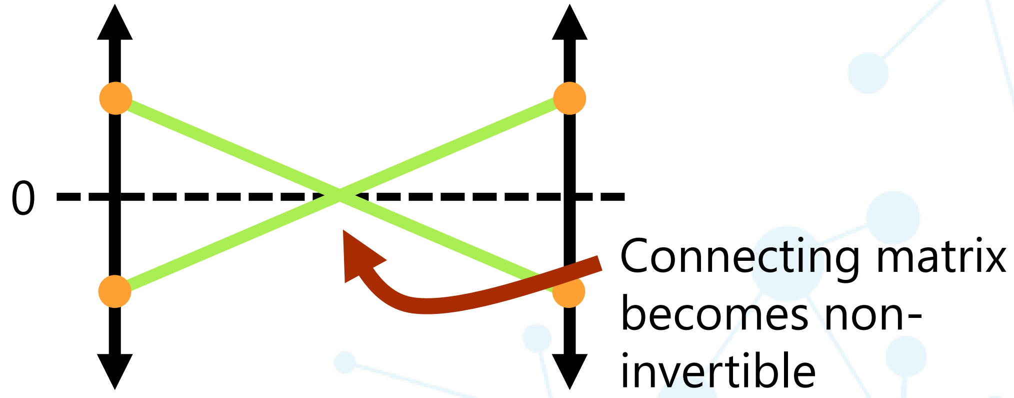
Pfaffian — $\mathrm{Pf}[T] = \alpha_1 \alpha_2 \cdots \alpha_n$

Determinant — $\det[T] = \mathrm{Pf}[T]^2$

If we want to change $\mathrm{sign}\big[\mathrm{Pf}[T]\big]$

while preserving $T^{\top} = -T$

$$\begin{bmatrix} \ddots & & & \\ & 0 & \alpha_j & \\ & -\alpha_j & 0 & \\ & & & \ddots \end{bmatrix} \to \begin{bmatrix} \ddots & & & \\ & 0 & -\alpha_j & \\ & \alpha_j & 0 & \\ & & & \ddots \end{bmatrix}$$



Classifying fragile topology via matrix homotopy

$$\left(WHW^\dagger\right)^\top = WHW^\dagger \qquad \sigma_y^\top = -\sigma_y$$
$$\left(WXW^\dagger\right)^\top = -WXW^\dagger \qquad \sigma_x^\top = \sigma_x$$
$$\left(WYW^\dagger\right)^\top = -WYW^\dagger \qquad \sigma_z^\top = \sigma_z$$

Form a (nearly) skew-symmetric spectral localizer

$$L_{(x,y,E)}\left(WXW^\dagger, WYW^\dagger, WHW^\dagger\right) = \kappa\left(WXW^\dagger - x\right) \otimes \sigma_x + \kappa\left(WYW^\dagger - y\right) \otimes \sigma_z + \left(WHW^\dagger - E\right) \otimes \sigma_y$$

$$= \begin{bmatrix} \kappa\left(WYW^\dagger - y\right) & \kappa\left(WXW^\dagger - x\right) - i\left(WHW^\dagger - E\right) \\ \kappa\left(WXW^\dagger - x\right) + i\left(WHW^\dagger - E\right) & -\kappa\left(WYW^\dagger - y\right) \end{bmatrix}$$

At $(x, y) = (0,0)$, this spectral localizer is skew-symmetric

So can define the energy-resolved invariant

$$\zeta_E(X, Y, H) = \mathrm{sign}\left[\mathrm{Pf}\left[L_{(x,y,E)}\left(WXW^\dagger, WYW^\dagger, WHW^\dagger\right)\right]\right]$$

Invariant distinguishes systems based on what atomic limits they can be path continued to

$$\zeta_E \in \{-1,1\} \cong \mathbb{Z}_2$$

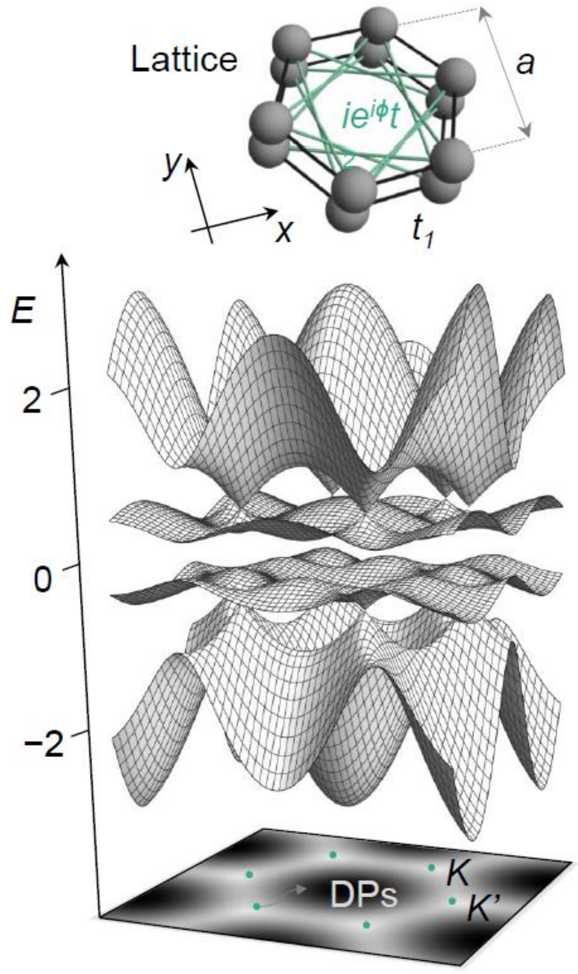
as expected

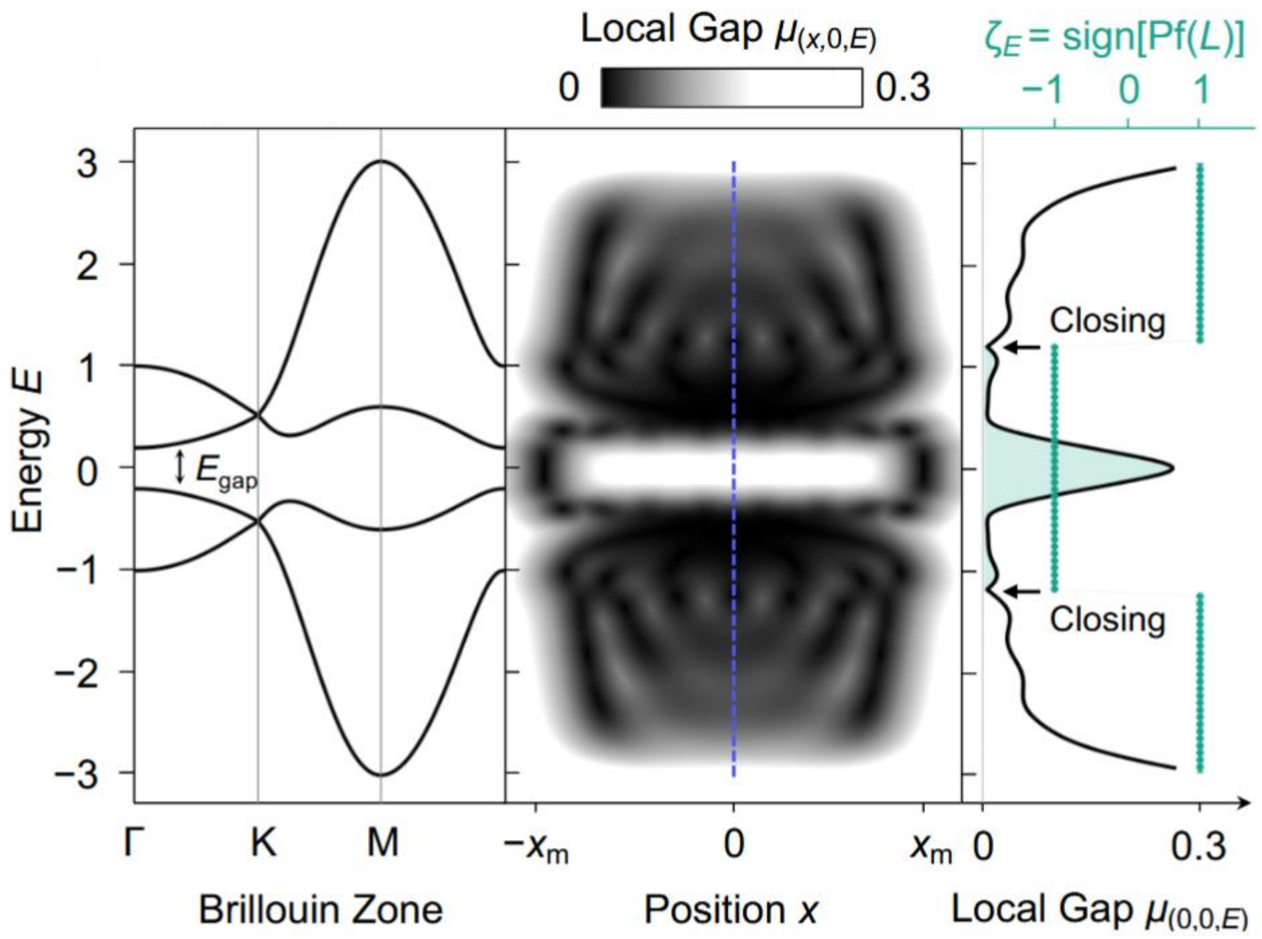
Same definition of topological protection

$$\mu^{\mathrm{C}}_{(x_1,\ldots,x_d,E)} = \sigma_{\min}\left[L_{(x_1,\ldots,x_d,E)}(X_1, \ldots, X_d, H)\right]$$

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

Classifying fragile topology via matrix homotopy





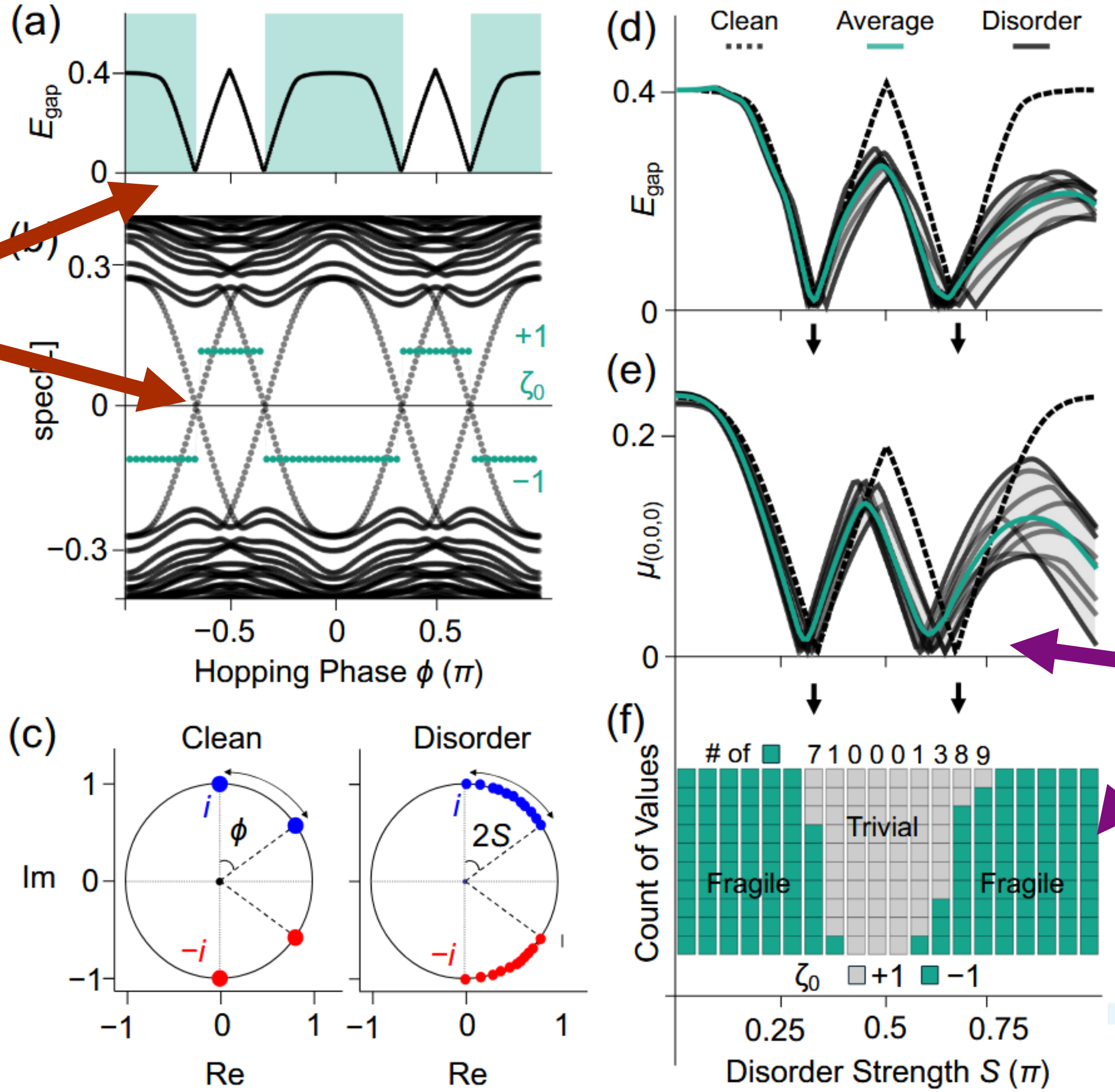

Ki Young Lee

Ahn, Park, and Yang, *Phys. Rev. X* **9**, 21013 (2019)

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

Re-entrant disorder-induced topology

Uniform changes in the in the NNN coupling phases induce phase transitions



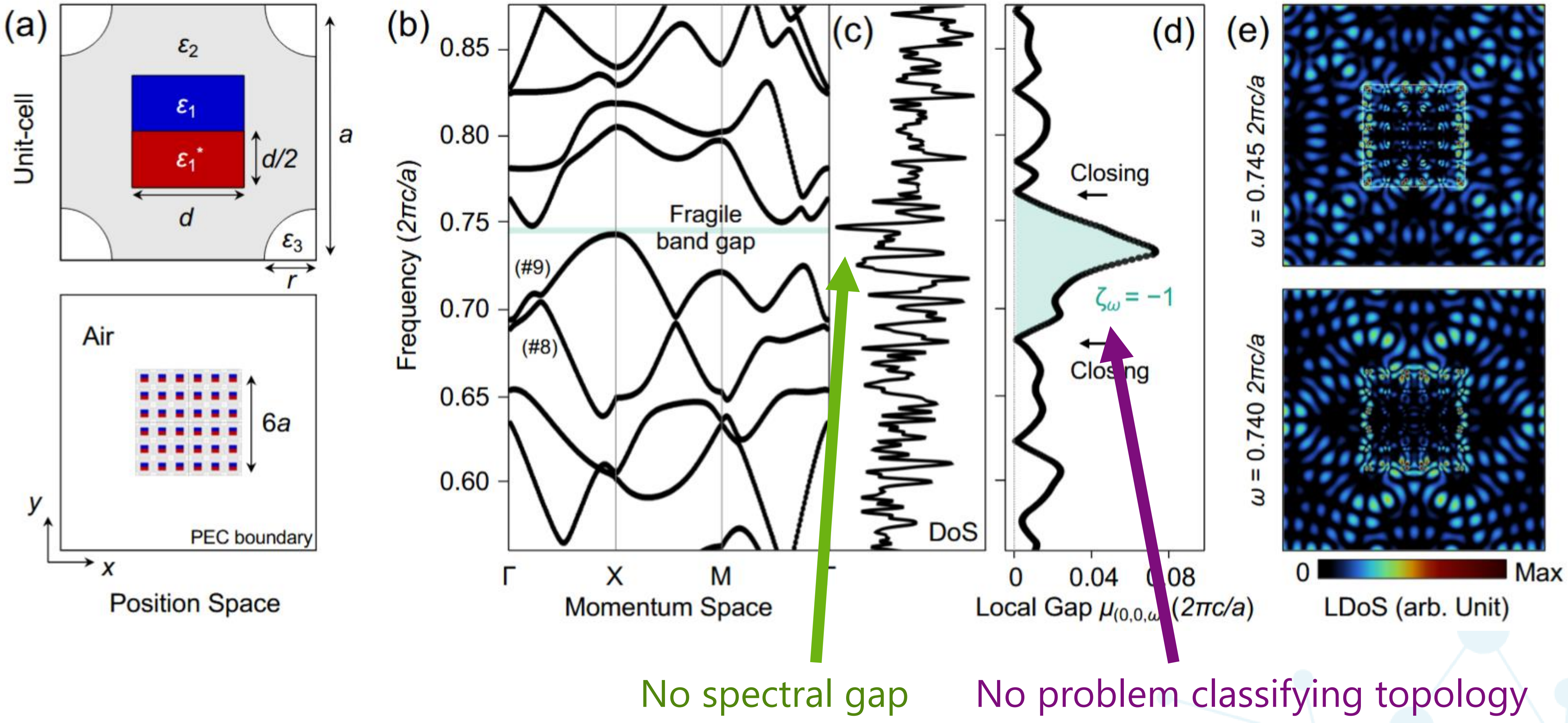
Disorder in NNN coupling phases also induces transitions

Stronger disorder yields

- Re-emergence of fragile phase

Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

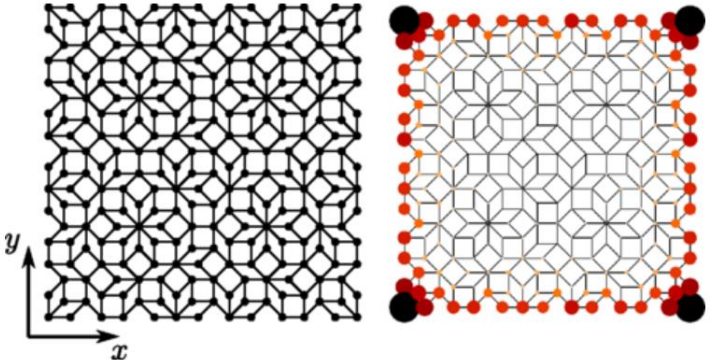
Gapless fragile photonic heterostructure



Lee, Wong, Vaidya, Loring, and AC, *Phys. Rev. Res.* **7**, 033235 (2025)

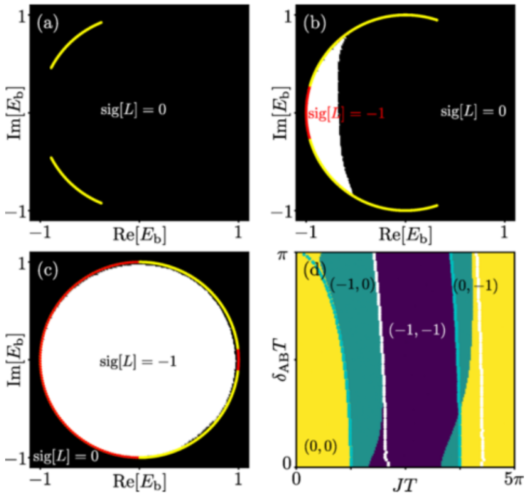
Useful in theory

Quasicrystals



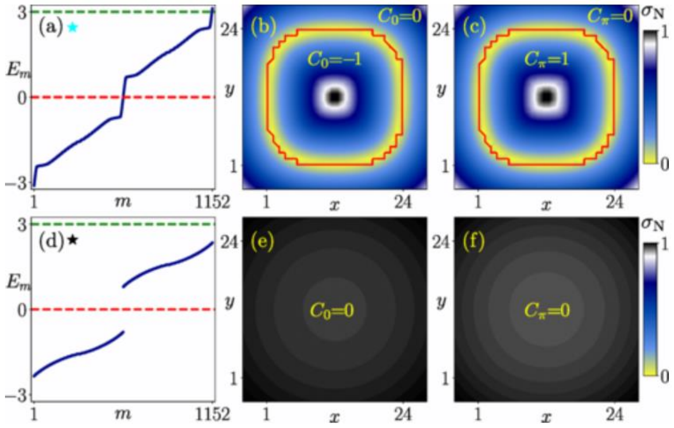
Fulga, Pikulin, and Loring, *Phys. Rev. Lett.* **116**, 257002 (2016)

Non-Hermitian skin-effect



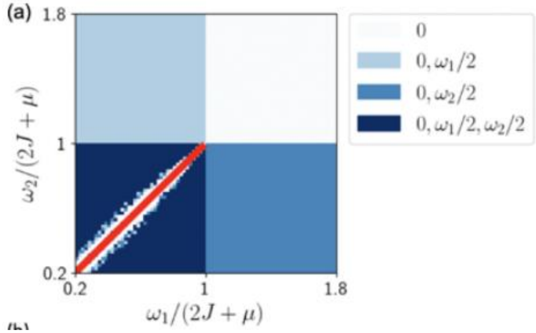
Liu and Fulga, *Phys. Rev. B* **108**, 035107 (2023)

Floquet topology (inc. anomalous)



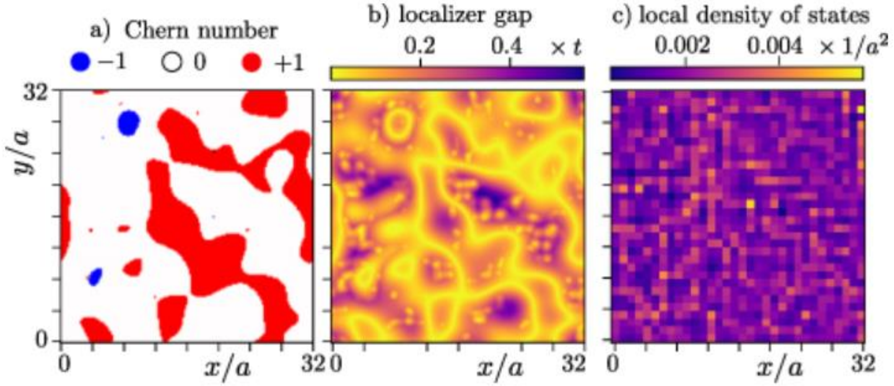
Ghosh, Arouca, and Black-Schaffer, *Phys. Rev. B* **110**, 245306 (2024)

Time-quasiperiodic Majoranas



Qi, Na, Refael, and Peng, *Phys. Rev. B* **110**, 014309 (2024)

Disordered topological superconductors



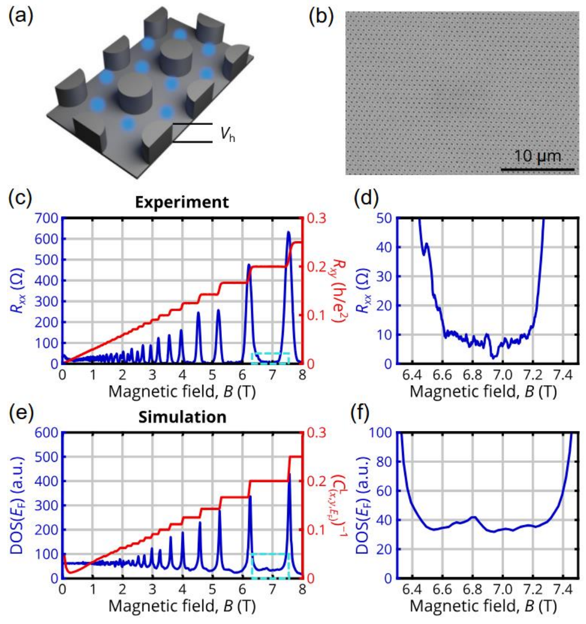
Zakharov, Fulga, Lemut, Tworzydło and Beenakker, *New J. Phys.* **27**, 033002 (2025)

Physics-based proof for invariant equivalence

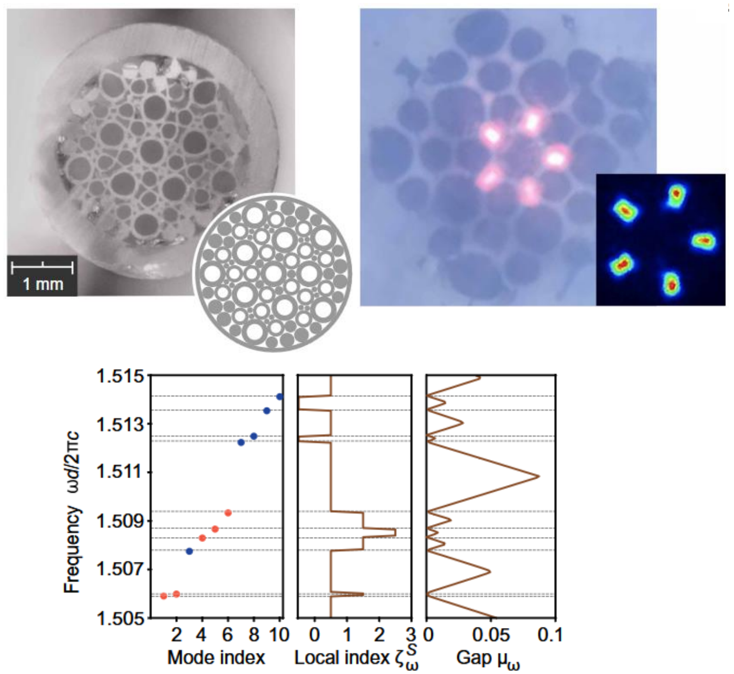
$$I_{\mathrm{SL}} = \frac{1}{2}\,\mathrm{Sig}(\hat{L}) = \frac{1}{2}\,\mathrm{Tr}'(\hat{L}_{\mathrm{F}}) = \frac{1}{2}\,\mathrm{Tr}'(\hat{L}(\hat{L}^2)^{-1/2})$$

Jezequel, Bardarson, and Grushin, arXiv:2508.00214

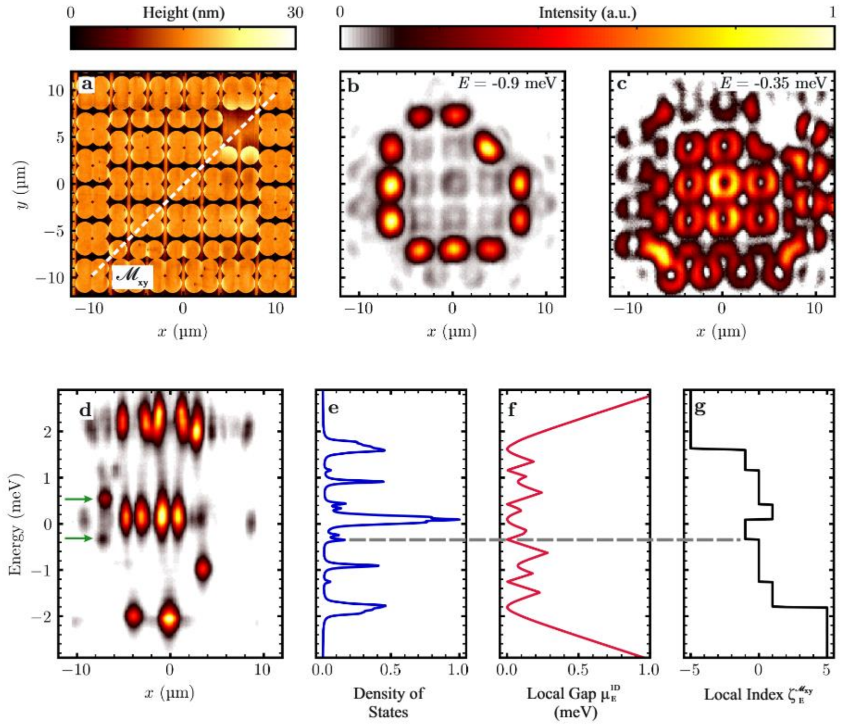
Useful in experiments



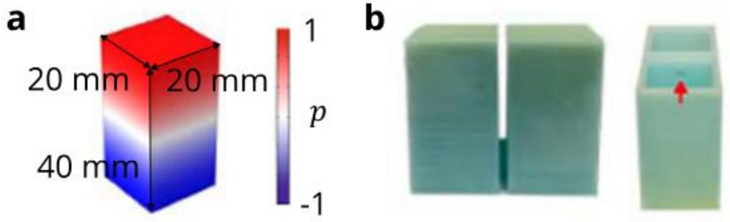
Spataru, Pan, and AC, *Phys. Rev. Lett.* **134**, 126601 (2025)



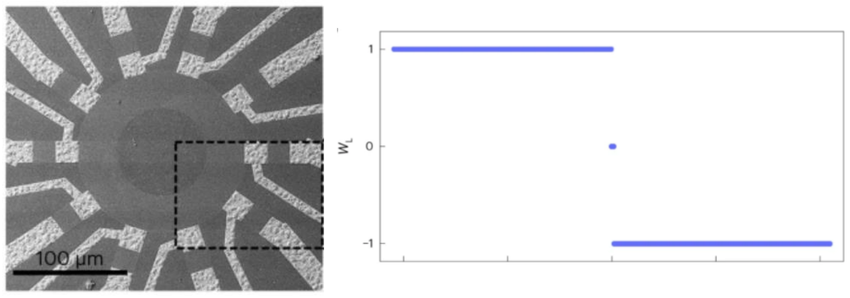
Zhu et al. (Chong group), *Sci. Adv.* **11**, eady1476 (2025)



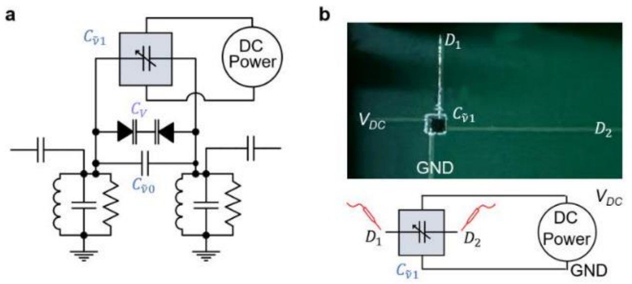
Düreth et al. (Klembt group), arXiv:2511.13958



Cheng et al. (Prodan group), *Nat. Commun.* **14**, 3071 (2023)



Ochkan et al. (Fulga group), *Nat. Phys.* **20**, 395 (2024)



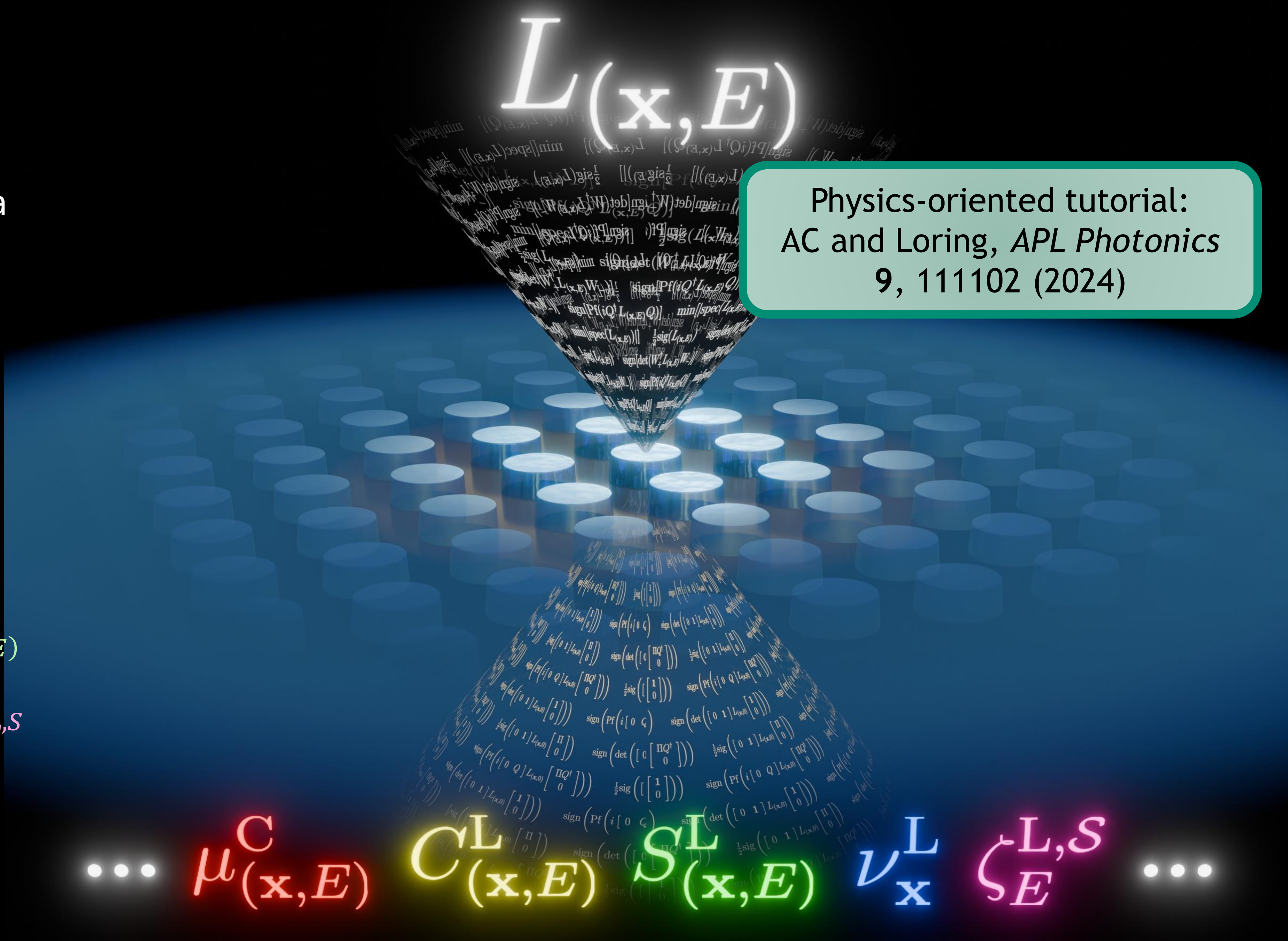
Bai et al. (Xiao group), arXiv:2509.04837

A "mathematical SEM"

If $\mu^{\mathrm{C}}_{(\mathbf{x},E)}$ is small – system has a nearby state

If $\mu^{\mathrm{C}}_{(\mathbf{x},E)}$ is large – the local topological phase is robust

- Can be classified with
  - Chern number $C^{\mathrm{L}}_{(\mathbf{x},E)}$
  - Quantum spin Hall $S^{\mathrm{L}}_{(\mathbf{x},E)}$
  - Winding number $\nu^{\mathrm{L}}_{\mathbf{x}}$
  - Crystalline topology $\zeta^{\mathrm{L},S}_{E}$
  - etc…



Physics-oriented tutorial:
AC and Loring, *APL Photonics* **9**, 111102 (2024)